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Quantum Hall effect in three-dimensional layered systems

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Using a mapping of a layered three-dimensional system with significant interlayer tunneling onto a spin Hamiltonian, the phase diagram in the strong magnetic-field limit is obtained in the semiclassical approximation. This phase diagram, which exhibits a metallic phase for a finite range of energies and magnetic fields, and the calculated associated critical exponent $\nu=4/3$ agree excellently with existing numerical calculations. The implication of this work for the quantum Hall effect in three dimensions is discussed.
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The quantum Hall effect is one of the hallmarks of two-dimensional electron systems.[1,2] The possibility of the occurrence of the quantum Hall effect in three dimensions was explored rather early,[3] and precursors of the quantum Hall effect were observed in some three-dimensional systems.[4] The existence of well quantized Hall plateaus was, however, demonstrated only in three-dimensional layered semiconductors with significant interlayer coupling.[5] These layered systems have attracted significant theoretical interest recently, due to the proposed existence of a metallic phase for a finite range of energies or magnetic fields,[6] and a new ''chiral'' two-dimensional metallic phase on the surface.[7] The existence of such a metallic phase at the surface was recently confirmed experimentally in measurements of the vertical conductance (σ_{zz}).[8]

In this work we use a mapping of the three-dimensional layered structure onto a two-dimensional spin Hamiltonian. Using a semiclassical description we derive the phase diagram[6] and obtain the critical exponent ν, describing the divergence of the localization length ξ, as one approaches the transition from the insulating side $\xi\sim|E-E_c|^{-\nu}$, or $\xi\sim|B-B_c|^{-\nu}$, where E_c and B_c are the critical energy and magnetic field, respectively. The derived critical exponent $\nu=4/3$ agrees excellently with existing numerical data, $\nu=1.35\pm0.15$, obtained both for a layered system and three-dimensional tight-binding model,[9] and $\nu=1.45\pm0.25$, obtained in Ref. 6 from a layered network model.[10]

We start with the Hamiltonian describing a spinless electron in a system of N coupled two-dimensional layers (see Fig. 1),

$$\mathcal{H}=\sum_{i}^{N}\left[(\mathbf{p}_i-e\mathbf{A}_i/c)^2/2m+V_i(x,y)+T_{i,i+1}(x,y)\right], \quad (1)$$

where x and y are coordinates in the plane and the summation is over the layers. The first and second terms in the brackets describe the kinetic and potential energies within a layer, while the third term describes the hopping between adjacent layers, which may depend on the position in the plane. The layer potentials are assumed to be independently distributed with zero mean.

We now associate with the electron a spin index that corresponds to the layer index in Eq. (1). The interlayer tunneling will now correspond to spin raising and lowering operations. In order to describe the different potential landscape in each layer, we add a random S_z term to the Hamiltonian that now describes a spin $S\,[=(N-1)/2]$ electron moving in two dimensions,

$$\mathcal{H}=(\mathbf{p}-e\mathbf{A}/c)^2/2m+U(x,y)+\Delta U(x,y)S_Z+t(x,y)S_+ +t^*(x,y)S_-. \quad (2)$$

The second term describes a random potential independent of the spin (layer) index. The third term accounts for the different potentials for the different spin direction, by a random shift of the potential between adjacent layers (at each point of the plane). Thus at each point the electron sees a different potential in each layer (or for each spin direction). Since the shift $\Delta U(x,y)$ is random in sign and in magnitude, the average potential in each layer is the same.[11]

The Hamiltonian (2) can now be simply written as

$$\mathcal{H}=(\mathbf{p}-e\mathbf{A}/c)^2/2m+U(x,y)+\frac{1}{S}\mathbf{S}\cdot\mathbf{H}(x,y), \quad (3)$$

namely, a spin-S electron moving in two dimensions under the influence of a random potential and a random magnetic field (coupled to its spin). The advantage of this representation is that one can try to generalize methods that worked for the two-dimensional case, in the absence of a random field, to include the effects of the field. In the following we will concentrate on the large (uniform) magnetic-field limit, where the kinetic energy is quenched and one may treat the electrons semiclassically. In the absence of the random field the electron moves along equipotential lines. As is well known in this case,[12] electrons with too small an energy will be trapped around potential valleys, while for too high an energy they will be trapped around potential hills. There is a

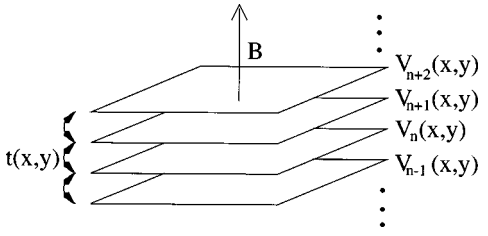

FIG. 1. The system studied in this work.

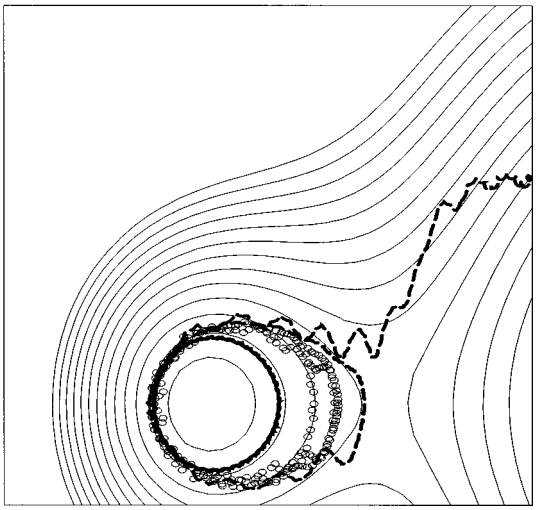

FIG. 2. The classical trajectories of an electron in a strong magnetic field. Without random field, the electron follows equipotential lines (solid curves); with increasing random field it explores a larger portion of the potential energy landscape (circles), until for a large enough random field, it can go through the saddle point (broken line).

single ''critical'' energy where the electron trajectory percolates through the system. This corresponds to the quantum Hall transition, where there is a single energy (at the center of the Landau level in case of symmetrically distributed random potentials) where states are extended.

In the present case, in the same strong magnetic-field limit, it is the total energy—the potential energy plus the spin energy (due to the random field)—that is conserved. Thus, as the electron rotates its spin along the trajectory, it exchanges energy between the potential energy and the spin energy, such that the total is conserved. The range of potential energies accessible by the electron has a width $\Delta \equiv 2H_R$, where H_R is the typical amplitude of the random field. Consequently, even if the electron does not have the correct (critical) potential energy to percolate through the system to begin with, it can still do that as long as its total energy is within H_R of the critical energy.

An example is depicted in Fig. 2. The classical equations of motion for the Hamiltonian (3) with $U(x,y)$ corresponding to two impurities (the equipotential lines appear as thin solid curves) were integrated. In the absence of a random field (a solid thick curve), the electron follows a single equipotential line, with superimposed cyclotron oscillations, and is trapped around one impurity. With an increasing random field the electron explores a larger portion of the potential energy landscape (see, e.g., the trajectory denoted by circles), until, for a large enough random field (broken line), the electron can go through the saddle point and percolate away. In the original layered system, this process corresponds to the possibility of the electron tunneling to a different layer and drifting along a different potential line (with the same potential energy). Thus as the energy is increased, before percolation occurs in a single layer, there will be a percolating path consisting of equipotential lines in different layers, connected by interlayer tunneling events.

Since the random magnetic-field amplitude $H_R \sim \sqrt{t^2+(\Delta U)^2}$, one expects a region of extended states that

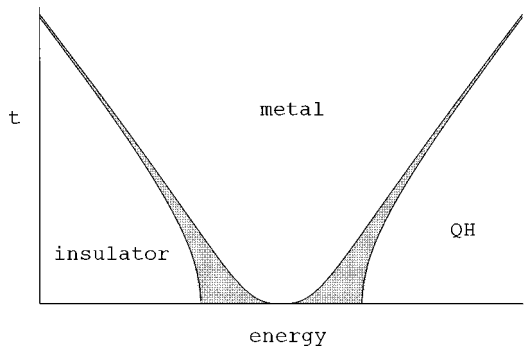

FIG. 3. Phase diagram of the layered system. For finite interlayer tunneling t there is a finite range of energies where extended states exist. The shaded region is where the wave functions are localized, but the classical trajectories are extended (see text).

increases with t, leading to the phase diagram depicted in Fig. 3. For any finite t there exists a finite range of energies (or magnetic fields) where the system is metallic. Accordingly, even at $T=0$ the transition between Hall plateaus will not be sharp, but rather occur in a finite range of magnetic fields or gate voltages.

Interestingly, in the present semiclassical description such a metallic phase will occur even for an infinitesimal tunneling matrix element t. The reason is that once $t \neq 0$ the electron can, in principle, rotate its spin (tunnel between layers) and explore the whole energy range allowed by conservation of total energy. We know, however, that quantum mechanically, for small enough tunneling matrix element, the electron will be localized in spin space and the range of potential energies available (i.e., the width of the metallic region in phase space) will be much smaller than one expects classically, going to zero as $t \to 0$.[13] Thus, there is a region in the phase diagram (the shaded part of Fig. 3), where the electron is localized quantum mechanically, but its classical trajectory is extended. The derived phase diagram (Fig. 3) agrees with the phase diagram established numerically by Chalker and Dohmen.[6]

We now turn to the critical behavior. For the two-dimensional quantum Hall problem Mil'nikov and Sokolov[14,15] used the following argument to predict the critical exponent. In the classical description, away from the critical energy E_c, the electron is confined to a percolation cluster of typical size ξ_p, the percolation coherence length. Near the threshold $\xi_p \sim |E_c - E|^{-\nu_p}$, where $\nu_p = 4/3$ is the two-dimensional percolation exponent. As one approaches the transition the clusters approach each other near saddle points of the potential energy landscape. While classically the electron cannot move from one cluster to another, quantum mechanically it can tunnel through the potential barrier. If the electron energy E is close enough to the transition, the potential barrier is close to parabolic and the tunneling probability through such a saddle point is proportional to $\exp[-(E_c - E)]$. The number of such saddle points through which tunneling occurs in a system of length L is typically L/ξ_p. Since the transmission coefficient is multiplicative, the conductance (or the tunneling probability) through the whole system is

$$\sigma_{2D} \sim [e^{-(E_c - E)}]^{L/\xi_p} \equiv e^{-L/\xi_{2D}}, \tag{4}$$

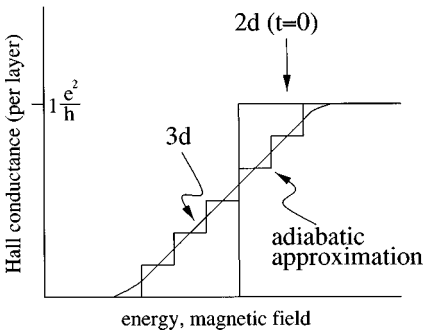

FIG. 4. The change in the Hall conductance at the transition. For $t=0$ there is a two-dimensional behavior (a single step). For $t\neq 0$, then in the adiabatic approximation one finds a series of smaller steps, of the number of layers (see text). In the three-dimensional limit, as the number of layers increases, one expects a smooth transition between the quantized values (a metallic region).

with $\xi_{2D}\sim(E_c-E)^{-\nu_{2D}}$ and $\nu_{2D}=\nu_p+1=7/3$. The best numerical estimate of the critical exponent $\nu_{2D}=2.35\pm 0.02$,[16] which is supported by experimental data,[17] has a surprisingly excellent agreement with the result of the above argument, especially in view of the crudeness of the argument.

This argument can be generalized to the present problem,[15] as it is also expressed in terms of a two-dimensional Hamiltonian. In the presence of interlayer tunneling (random field), the only difference between the present problem and the two-dimensional problem is the fact that the critical energy E_c is not equal to the potential energy of the saddle point, but is H_R away from it. Thus

$$\sigma_{3D}\sim[e^{-H_R}]^{L/\xi_p}\equiv e^{-L/\xi_{3D}}, \tag{5}$$

with $\xi_{3D}\sim(E_c-E)^{-\nu}$ and $\nu=\nu_p=4/3$. One finds the surprising result that the critical exponent for the quantum three-dimensional problem is equal to the two-dimensional classical percolation exponent. This result is in excellent agreement with existing numerical estimates, $\nu=1.35\pm 0.15$, obtained both for a layered system and three-dimensional tight-binding model,[9] and $\nu=1.45\pm 0.25$, obtained[6] from a layered network model.[10]

Consider now the Hall conductance σ_{xy}. If the interlayer tunneling t is equal to zero, the system is a collection of N independent two-dimensional layers, all with the same critical energy. Thus σ_{xy} will jump by e^2/h in all layers simultaneously (see Fig. 4), i.e., it will have a single step of height Ne^2/h (which corresponds to a conductance per layer or conductivity of e^2/h). For finite t (or finite random field) the situation is quite different. To see this we first carry out a local $SU(N)$ gauge transformation in spin space, to rotate the spin by a unitary matrix $U(x,y)$, such that the z direction always lies in the direction of the random field. This exact transformation maps the Hamiltonian (3) onto the equivalent Hamiltonian[18]

$$\mathcal{H}=(\mathbf{p}-e\mathbf{A}/c-i\hbar U^{\dagger}\nabla U)^2/2m+U(x,y)+\frac{1}{S}S_z|H(x,y)|. \tag{6}$$

If the potential energy and the interlayer tunneling vary slowly in space, one may apply the adiabatic approximation.[19] In this approximation one neglects the additional $U^{\dagger}\nabla U$ term in the parentheses, and the Hamiltonian can be trivially diagonalized in spin space. The random field serves as an additional potential energy, which is different for each spin direction (and its average is proportional to $H_R S_z$). Consequently, in this approximation one expects N separate transitions, each of the two-dimensional type (see Fig. 4). (Note that these transitions are not related to the different layers, but rather to different coherent superpositions of the wave functions in different layers). Since the separate transitions can only be resolved for energies smaller than Δ/N, one expects in this case a crossover from a three-dimensional critical behavior for $|E-E_c|>\Delta/N$ to a two-dimensional critical behavior for $|E-E_c|<\Delta/N$ (the two-dimensional behavior can only be seen for temperatures smaller than Δ/N),

$$\xi=A_1\epsilon^{-\nu_{2D}}f(\epsilon),\quad f(\epsilon)\rightarrow\begin{cases}1 & \epsilon\ll 1\\ A_2\epsilon^{\nu_{2D}-\nu_{3D}} & \epsilon\gg 1,\end{cases} \tag{7}$$

with $\epsilon\equiv(E-E_c)/(\Delta/N)$. Thus, the effective exponent ν will cross over from its three-dimensional ($\sim 4/3$) to the two-dimensional value ($\sim 7/3$), as one gets closer to the critical point from the insulating side. Interestingly, for the case $\nu_{2D}=7/3$ and $\nu_{3D}=4/3$ the scaling function $f(\epsilon)$ may be analytic. This crossover can be studied via the critical behavior of the conductance [Eqs. (4) and (5)], or by that of $d\sigma_{xy}/dB$.[20]

In the adiabatic approximation there is a zero-temperature metallic phase only in the true three-dimensional limit ($N\rightarrow\infty$), which is the classical limit ($S\rightarrow\infty$) of the spin problem (Fig. 4). As nonadiabaticity [the additional term in the parentheses in Eq. (6)] is switched on, the different spin states that were the eigenstates of the system in the adiabatic limit get coupled. It is not clear if this coupling will smear out the separate transitions even for a finite number of layers. It is known that there may occur transitions between the expected adiabatic behavior to a different behavior (as a function of, e.g., the tunneling matrix element), even for the two-layer problem,[21] and it remains to be seen if such a deviation from the adiabatic limit will also occur for a finite number of layers. We hope that this work will motivate further studies in this direction.

To conclude, we have used a mapping onto a two-dimensional spin Hamiltonian to describe the physics of the quantum Hall effect in three-dimensional layered systems. This mapping was used mainly for conceptual reasons, in order to allow us to extend methods applied in the traditional two-dimensional quantum Hall systems to the present case. The arguments presented here, however, could be directly applied to the original three-dimensional system, and thus none of the results of this paper depends on the particular form of the spin Hamiltonian. For example, in the three-dimensional layered system, the potential and the hopping part of the Hamiltonian [Eq. (1)] can be recast in the form of a position-dependent $N\times N$ matrix. Diagonalizing this matrix locally and carrying out a unitary local rotation in layer space, will lead to a Hamiltonian of the form (6), and to all the results of the last section. Similar arguments can be made to derive the phase diagram and the critical exponent.

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[11] Note, however, that the Hamiltonian (2) has less degrees of freedom than Hamiltonian (1), since at each point in the plane the potential in spin space forms a ladder, spaced by energy $\Delta U(x,y)$. Nevertheless, if one imagines coarse graining this Hamiltonian, then at each iteration more and more frequencies will appear and instead of an equally spaced ladder, one will have now quasiperiodicity, with increasing number of frequencies. Thus one may expect that under such renormalization, the Hamiltonian (2) will flow to the same fixed point as Eq. (1), a fact that is borne out by the phase diagram and the critical exponent, which seem to be identical. Similarly, the weak dependence of the tunneling on the layer number is expected not to be a relevant perturbation.

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[15] Note that the previous reference failed to include the Aharonov-Bohm phases along the electron trajectories. In the present system this is not a problem, as the Aharonov-Bohm phase becomes irrelevant as the spin increases [Y. Meir, Y. Gefen, and Y. Aharonov, in *Quantum Coherence in Mesoscopic Systems*, edited by B. Kramer (Plenum, New York, 1991)]. The physical reason is that the probability that two trajectories will end up with the same spin (i.e., on the same layer) goes to zero with increasing spin.

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