Magnetohydrodynamics Using Path or Stream Functions

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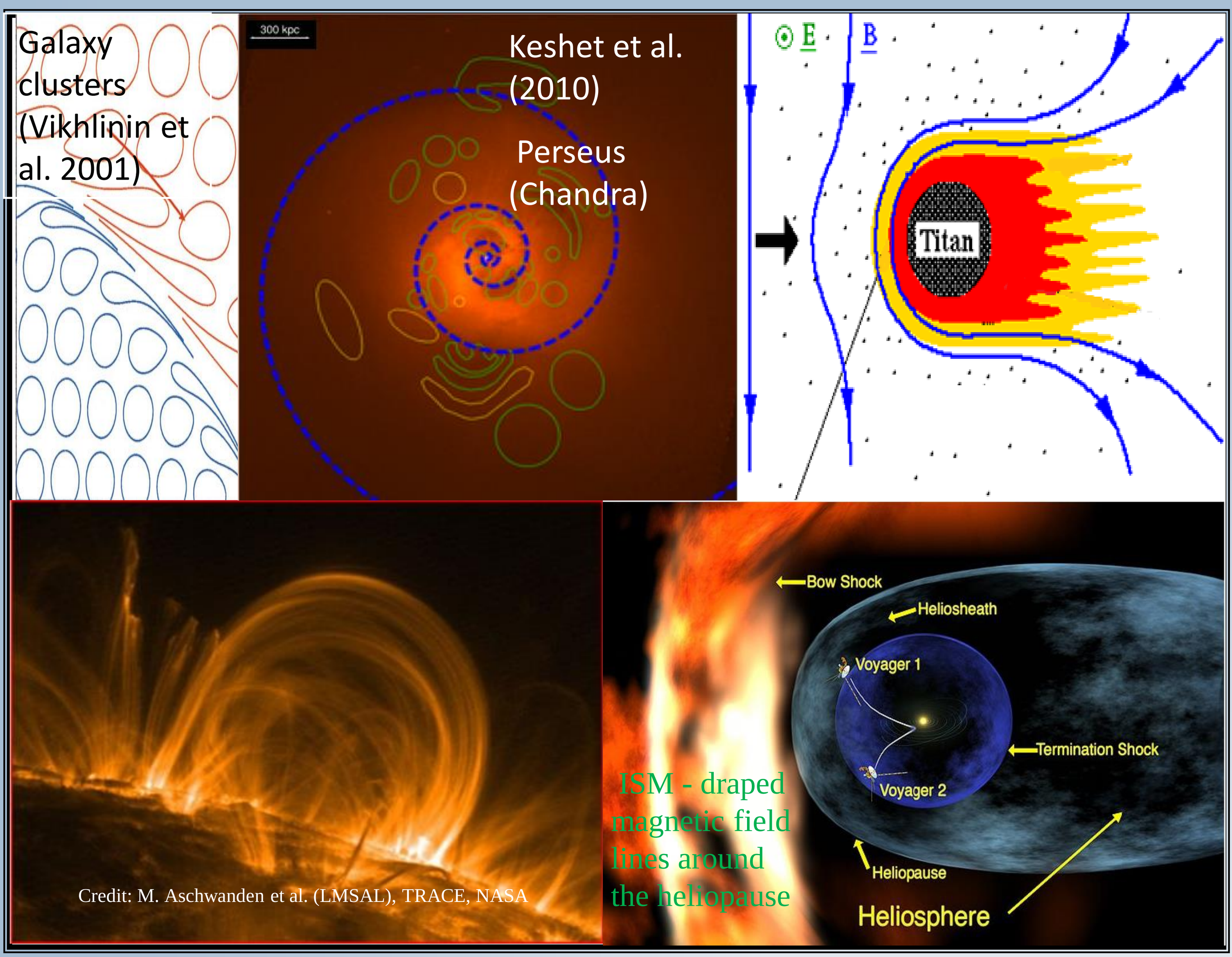
MAGNETIC FIELDS IN ASTRONOMICAL SYTEMS



IDEAL EM FIELD – AN ANALYTIC EXPRESSION IS NEEDED

Eulerian analysis

Gauss' law: $\nabla \cdot \boldsymbol{B} = 0$;

Continuity: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$;

Convection: $\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = 0$;

$$\frac{d}{dt}\left(\frac{\boldsymbol{B}}{\rho}\right) = \left(\frac{\boldsymbol{B}}{\rho}\cdot\nabla\right)\boldsymbol{v}$$

Solution

Elsasser (1956)

$$\boldsymbol{B} = \frac{\rho}{\tilde{\rho}}\left(\tilde{\boldsymbol{B}}\cdot\tilde{\nabla}\right)\boldsymbol{x}(t)$$

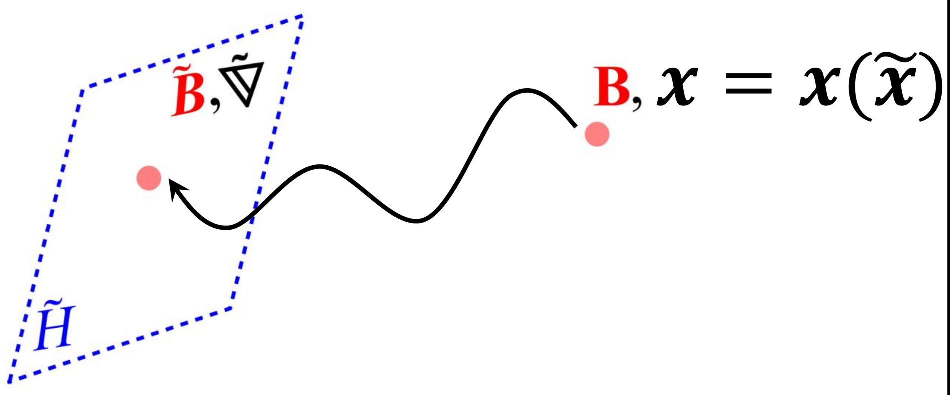
Ohm' law: $\boldsymbol{E} + \frac{v}{c}\times \boldsymbol{B} = 0$;

Navier-Stokes: $\rho\frac{dv_i}{dt} = \rho\frac{\partial\Phi}{\partial x_i} - \frac{\partial}{\partial x_i}\left(P_T + \frac{B^2}{8\pi}\right) + \frac{(B\cdot\nabla)B_i}{4\pi}$;

State equation.

(Known $\boldsymbol{v}$ is needed in advance in order to find $\boldsymbol{B}$.)

Non Local Expression!
One must first integrate over the flow to compute the mapping of $\boldsymbol{x}$ on $\tilde{H}$
(a hypersurface on which $\boldsymbol{B}$ is known)



Solutions:

- Numerical
- Semi-analytic
  - simplified surface representing bow shocks
  - (Corona-Romero & Gonzalez-Esparza 2013)
    or the Heliopause (Röken et al. 2014)
- Analytical:
  e.g. incompressible potential flow around a sphere
  (Dursi & Pfrommer 2008; Romanelli et al. 2014)

Typical strategy for finding $\boldsymbol{B}$ when $\boldsymbol{v}$ is known:

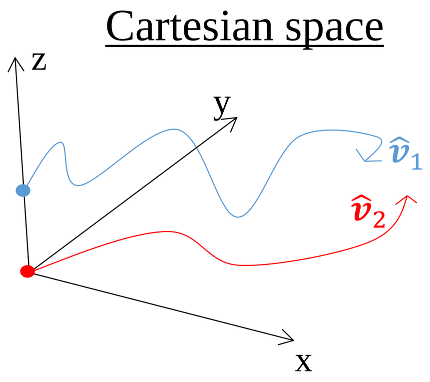
1. Forget about Elsasser's Equation (complicated to use)
2. Forget about time dependency (complicated)
3. Solve $\nabla \times (\boldsymbol{v}\times\boldsymbol{B}) = 0$ (difficult even for simple $\boldsymbol{v}$)
4. Use constant $\tilde{\boldsymbol{B}}$ (difficult enough)

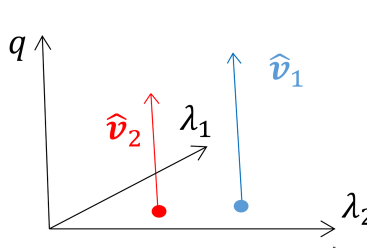
Isn't there a simpler way?

Lagrangian analysis

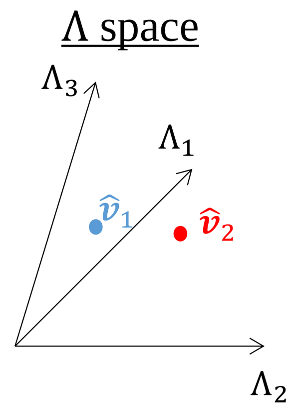
Since the magnetic field is "frozen" to the flow, we can use stream (λ) or path (Λ) functions (that describe the flow) in order to derive a local EM field expression.

Cartesian space — λ space — Λ space





$$\frac{\rho \boldsymbol{v}}{\tilde{\rho}\tilde{v}} = (\nabla\lambda_1\times\nabla\lambda_2) = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ \frac{\partial\lambda_1}{\partial x_1} & \frac{\partial\lambda_1}{\partial x_2} & \frac{\partial\lambda_1}{\partial x_3} \\ \frac{\partial\lambda_2}{\partial x_1} & \frac{\partial\lambda_2}{\partial x_2} & \frac{\partial\lambda_2}{\partial x_3}\end{vmatrix}$$



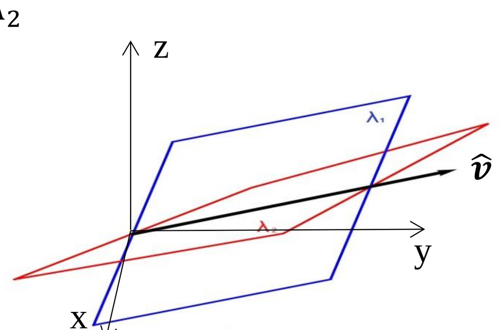
$$\frac{\rho}{\tilde{\rho}}\begin{pmatrix}1\\ v\end{pmatrix} = \frac{\rho}{\tilde{\rho}}(1+v) = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 & 1 \\ \frac{\partial\Lambda_1}{\partial x_1} & \frac{\partial\Lambda_1}{\partial x_2} & \frac{\partial\Lambda_1}{\partial x_3} & \frac{\partial\Lambda_1}{\partial t} \\ \frac{\partial\Lambda_2}{\partial x_1} & \frac{\partial\Lambda_2}{\partial x_2} & \frac{\partial\Lambda_2}{\partial x_3} & \frac{\partial\Lambda_2}{\partial t} \\ \frac{\partial\Lambda_3}{\partial x_1} & \frac{\partial\Lambda_3}{\partial x_2} & \frac{\partial\Lambda_3}{\partial x_3} & \frac{\partial\Lambda_3}{\partial t}\end{vmatrix}$$

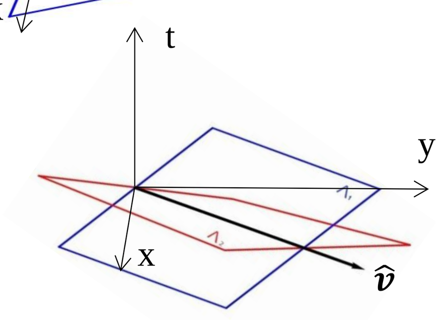
Path and Stream functions

- ❖ Exist as long as particle diffusion can be neglected (Yih 1957).
- ❖ Constant along the flow
- ❖ Exist when discontinuity surfaces are involved.

In order to describe the flow in 3D space we need 3 functions:

- In time dependent flows: $\Lambda_{\alpha=1,2,3}$.
- In steady flows: $\lambda_{\kappa=1,2}$ and an additional quantity φ (for example ρ or v).





Steady State:

$$\boldsymbol{B} = \frac{\rho}{\tilde{\rho}}\left[\tilde{B}_\kappa\frac{\partial \boldsymbol{x}}{\partial\lambda_\kappa} + \boldsymbol{v}\left(\frac{\tilde{B}_3}{\tilde{v}} - \tilde{B}_\kappa\frac{\partial\Delta t}{\partial\lambda_\kappa}\right)\right]$$

$$\boldsymbol{E} = \frac{\tilde{\rho}\,\tilde{v}}{\tilde{\rho}c}\left(\tilde{B}_2\nabla\lambda_1 - \tilde{B}_1\nabla\lambda_2\right)$$

$\tilde{B}_\kappa = (\boldsymbol{B}\cdot\nabla\lambda_\kappa)_{\tilde{H}}$ $\quad v_q = \frac{dq}{dt} = \boldsymbol{v}\cdot(\nabla q)$

$\tilde{B}_3 = \left(\frac{v}{v_q}\boldsymbol{B}\cdot\nabla q\right)_{\tilde{H}}$ $\quad \Delta t(\boldsymbol{x}) = \int_{\tilde{q}}^{q}\frac{dq'}{v_q(\lambda_\kappa, q')}$

Time dependent:

$$\boldsymbol{B} = \frac{\rho}{\tilde{\rho}}\tilde{B}_\alpha\left(\frac{\partial \boldsymbol{x}}{\partial\Lambda_\alpha} - \boldsymbol{v}\frac{\partial t}{\partial\Lambda_\alpha}\right)$$

$$\boldsymbol{E} = \epsilon_{ijk}\frac{\tilde{\rho}\tilde{B}_i}{\tilde{\rho}c}\left(\frac{\partial\Lambda_j}{\partial t}\right)\nabla\Lambda_k$$

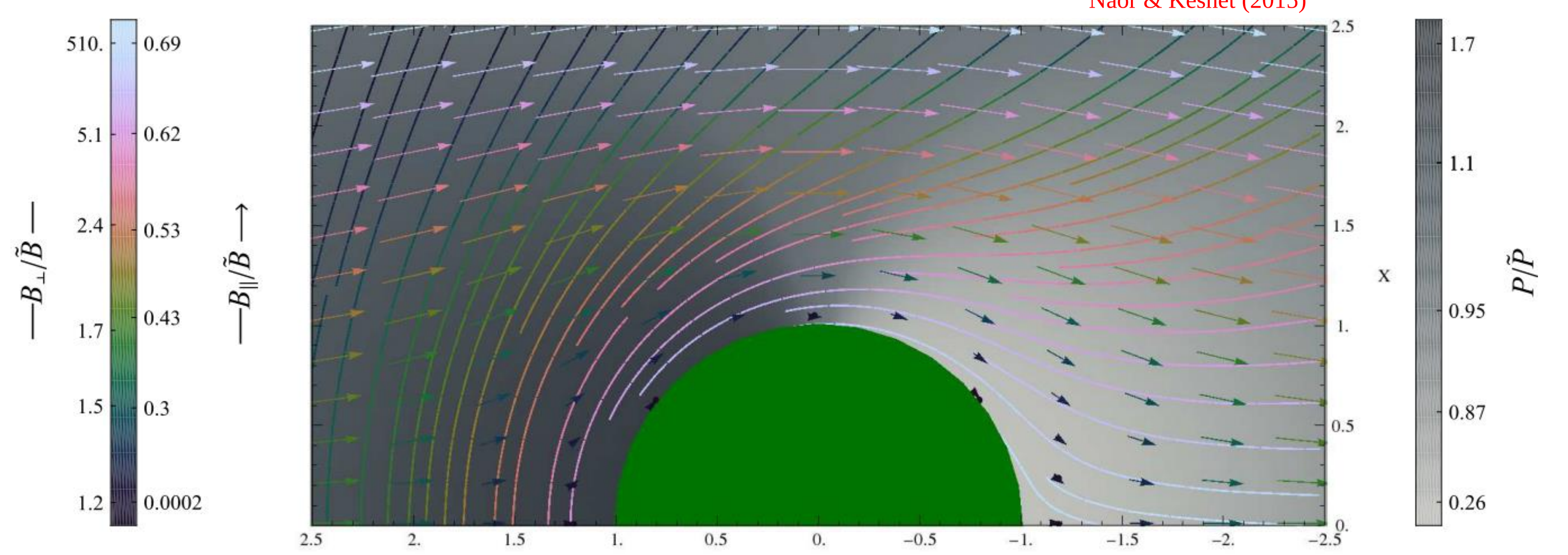
$\tilde{B}_\alpha = (\boldsymbol{B}\cdot\nabla\Lambda_\alpha)_{\tilde{H}}$

Fourth parameter, $Q(\boldsymbol{x},t)$, to parameterize space-time.

Applications

Steady incompressible Stokes flow:

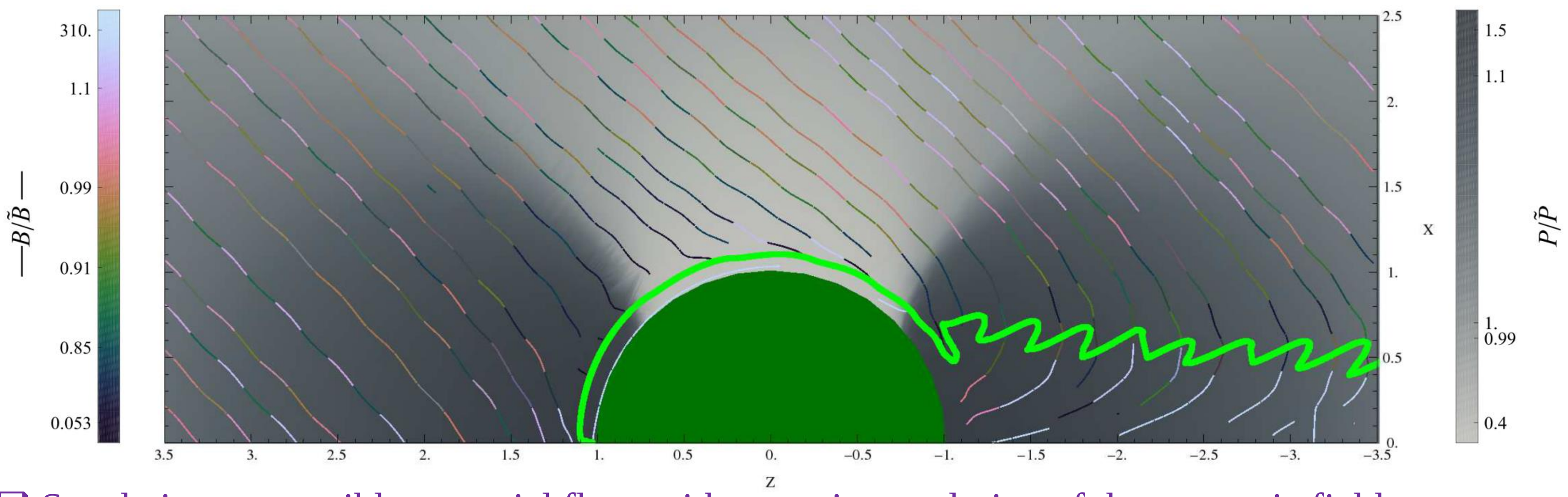
Naor & Keshet (2015)



$$0 = -\nabla\left(\frac{P}{\tilde{\rho}}\right) + \nu\nabla^2\boldsymbol{v} \qquad q = z\ ;\ \lambda_1 = R\phi\ ;\ \lambda_2 = \frac{\varrho^2}{2R}\left(1-\frac{R}{r}\right)^2\left(1+\frac{R}{2r}\right)$$

Inverse polarity reversal layer ($B_v = 0$):

Naor & Keshet (2015)



- ❑ Steady incompressible potential flow, with a passive evolution of the magnetic field

$$\nabla\cdot\boldsymbol{v} = 0\ ;\ \nabla\times\boldsymbol{v} = 0\ ;\ q = z\ ;\ \lambda_1 = R\phi\ ;\ \lambda_2 = \frac{\varrho^2}{2R}\left(1-\frac{R^3}{r^3}\right)$$

- ❑ $\tilde{\boldsymbol{B}}$ here is position dependent $\quad \tilde{\boldsymbol{B}} = B_0\left(\sin\theta_0,\ 0,\ \cos\theta_0\right) + \hat{\boldsymbol{x}}\zeta B_0\sin(\theta_0)\cos\omega\tilde{t}$
- ❑ The angle between $\tilde{B}_z$ and $\tilde{B}_x$ is $\theta_0 = 45°$ $\quad \tilde{t} = -\Delta t;\ \omega = 6\pi;\ \zeta = 0.25$

EM field Above Heliospheres and Magnetospheres:

$$\Phi = -\left(z + \frac{L_m^2}{r}\right)$$ The normalized potential of the Rankine half-body.

L_m is the distance between the object and the nose of the discontinuity.

$$\boldsymbol{B}_\Omega = \frac{\tilde{B}_\varrho}{\sqrt{f}}\begin{pmatrix} -\cos^3(\theta/2) \\ -\tilde{B}_\phi/\tilde{B}_\varrho \\ \left(1+\frac{\cos\theta}{2}\right)\sin(\theta/2)\end{pmatrix} + O\left(\sqrt{\tilde{\epsilon}_B}\right)$$

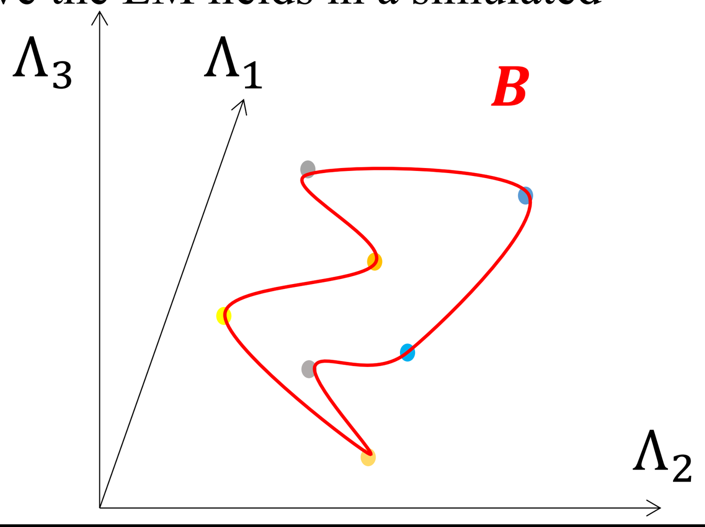
$$\lambda_1 = \breve{R}\phi \qquad \lambda_2 = \frac{1}{2\breve{R}}\left[\varrho^2 + 2L_m^2\left(\frac{z}{r}-1\right)\right] = \frac{\tilde{\varrho}^2}{2\breve{R}}$$

$$\boldsymbol{E}_\Omega = -\frac{2\tilde{B}_\phi\tilde{u}\sin^2(\theta/2)}{c\sqrt{f}}\begin{pmatrix} 1+\frac{\cos\theta}{2} \\ 0 \\ \frac{\cos^3(\theta/2)}{\sin(\theta/2)}\end{pmatrix} + O\left(\sqrt{\tilde{\epsilon}_B}\right)$$

$$f \equiv \frac{\tilde{\epsilon}_B}{1+\frac{1+4\cos\theta+3\cos2\theta}{16\tilde{\chi}}} \qquad \epsilon_B \equiv \beta_p^{-1} = \frac{B^2/8\pi}{P_{th}} \qquad \chi \equiv P_{th}/P_{ram}$$

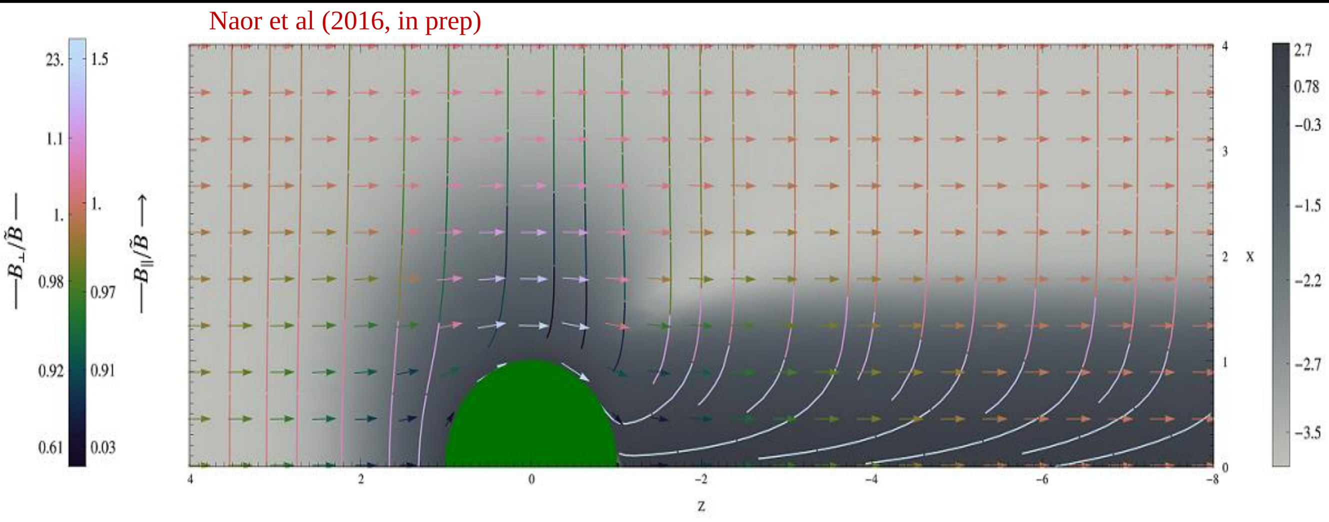
MHD simulations:

- Our analysis can be immediately incorporated into existing hydrodynamic codes that are based on stream or path functions (e.g. Pearson 1981, Beale 1993, Loh & Hui 2000, Hui 2007), in order to passively evolve the EM fields in a simulated flow.
- In a path functions prescription, the 3D EM fields are frozen onto the grid.
- The EM fields do not need to be evolved.
- Their back reaction on the flow can be easily computed (an efficient MHD simulation).



Strong magnetization in dense astronomical systems

Naor et al (2016, in prep)



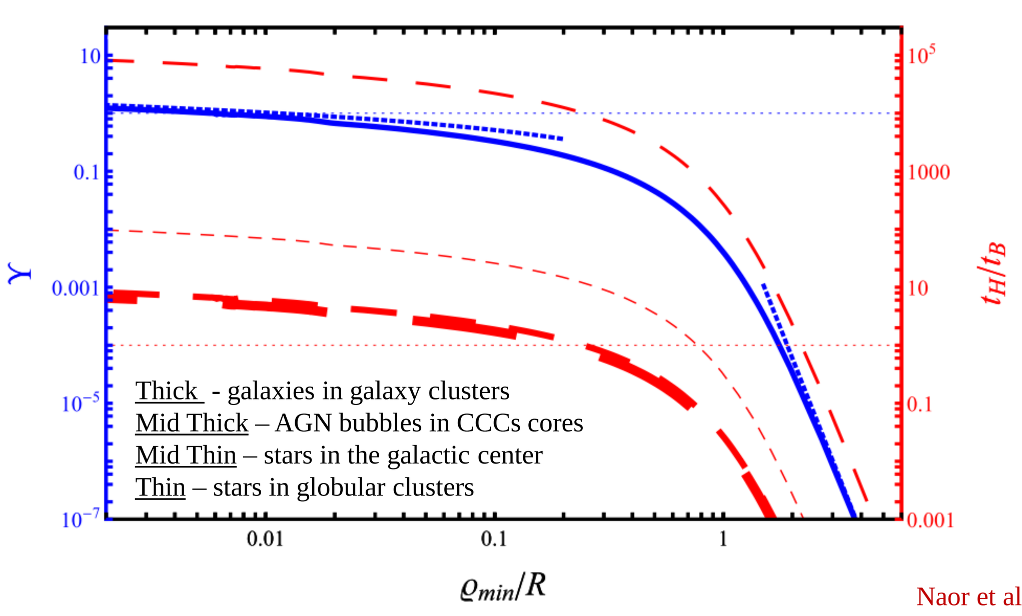
ΔB ≠ 0

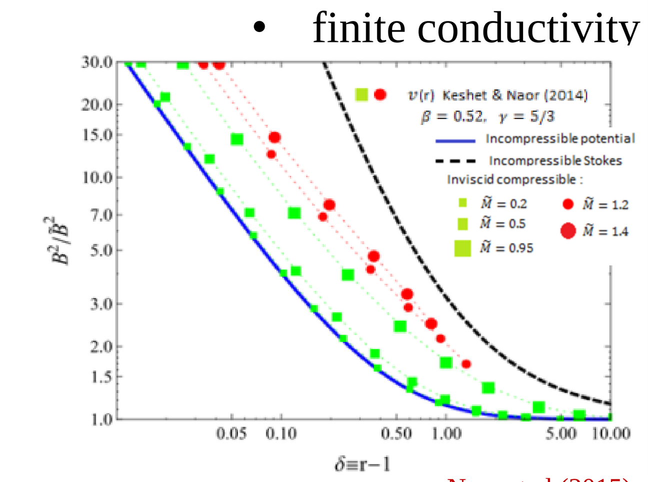
Maximal Timescale
Neglect:

- Viscosity
- Compressibility
- Time dependency
- Turbulence
- Non linear effects
- finite conductivity

$$t_B \sim \frac{L^3}{4N\Upsilon R^2\sigma}$$

$$\Upsilon = \begin{cases} -\frac{2}{9}\log\left(\frac{\varrho_{min}}{R}\right), & \varrho_{min} \ll R\ \ (Z<<-R) \\ \frac{5\times3^4\pi^2}{2^{16}}\left(\frac{\varrho_{min}}{R}\right)^{-10}, & R \ll \varrho_{min} \ll |z| \end{cases}$$





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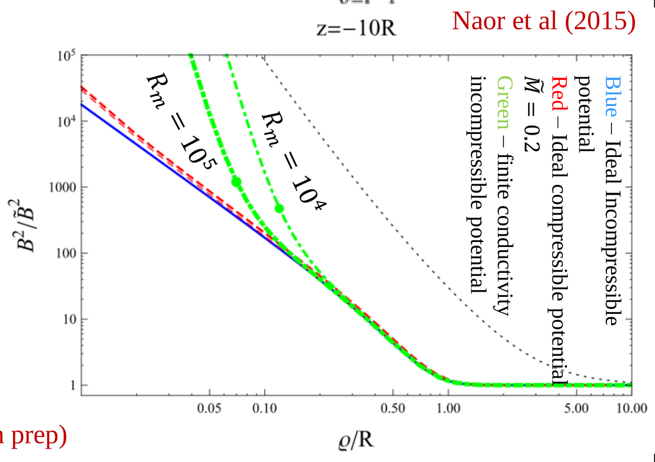

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