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The mesoscopic conductance of disordered rings, its random matrix theory and the generalized variable range hopping picture

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Abstract
The calculation of the conductance of disordered rings requires a theory that goes beyond the Kubo–Drude formulation. Assuming 'mesoscopic' circumstances the analysis of the electro-driven transitions shows similarities with a percolation problem in energy space. We argue that the texture and the sparsity of the perturbation matrix dictate the value of the conductance, and study its dependence on the disorder strength, ranging from the ballistic to the Anderson localization regime. An improved sparse random matrix model is introduced to capture the essential ingredients of the problem, and leads to a generalized variable range hopping picture.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Closed mesoscopic rings provide the ideal paradigm for testing the manifestation of quantum mechanical effects in the mesoscopic realm [1–4]. First measurements of the conductance of closed rings have been reported more than a decade ago [5], while more recently there is a renewed experimental interest motivated by high-precision measurements of individual rings [6, 7]. In a typical experiment, a collection of mesoscopic rings is driven by a time-dependent magnetic flux which creates an electro-motive-force (EMF). In what follows, we assume low-frequency dc *noisy* driving ($\omega \sim 0$) with power spectrum

$$\tilde{F}(\omega) = \varepsilon^2 \frac{1}{2\omega_c} \exp\left(-\frac{|\omega|}{\omega_c}\right) \equiv \varepsilon^2 \delta_\Gamma(\omega), \tag{1}$$

J. Phys. A: Math. Theor. **41** (2008) 262001   Fast Track Communication

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where ε is the rms value of the voltage and the cutoff frequency ω_c is small compared with any relevant semiclassical energy scale, but larger compared with the mean level spacing Δ.[3] Optionally, if we had assumed an interaction with a thermal bath, the role of ω_c would have been played by the level broadening or by the temperature [8]. In such a setup one expects the rate of energy absorption to be given by Joule's law $G\varepsilon^2$, where the coefficient G is defined as the 'conductance'.[4]

As in the linear response theory (LRT) analysis, we assume that the coherence time is much longer compared with the ballistic time, but smaller compared with the Heisenberg time $(1/\Delta)$. But our interest is in what we call *mesoscopic circumstances*. Namely, *we assume that the environmental-induced relaxation is a slow process, when compared with the EMF-driven transitions* (this is opposite to the LRT limit $\varepsilon \to 0$). Accordingly, we have to work within the framework of *semi-linear response theory* (SLRT) [9–11]:[5] this theory (see section 5) goes beyond the conventional framework of the Kubo formalism.

For diffusive rings the Kubo formalism leads to the Drude formula for G. A major challenge in past studies was to calculate the weak localization corrections [3] to the Drude result, taking into account the level statistics and the type of occupation [4]. These corrections are of order Δ/ω_c and accordingly do not challenge the leading-order Kubo–Drude result. It is just natural to ask what happens to the Drude result if the disorder becomes weak (ballistic case) or strong (Anderson localization case). In the latter case, there are two conflicting results for the noise ω_c dependence of G, both following Mott's work [12]. The question is whether to regard the noise as 'low-frequency driving' or as 'temperature'. On the one hand, on the basis of the Kubo formula, one expects a crossover from $G \sim \exp(-L/\ell_\infty)$ (where ℓ_∞ is the localization length), to the noise-dependent result $G \sim \omega_c^2 |\log(\omega_c)|^{d+1}$, where $d = 1$ for quasi-one-dimensional (1D) ring. On the other hand, on the basis of the variable range hoping (VRH) picture, one expects $G \sim \exp(-(\omega_0/\omega_c)^{1/d+1})$, where ω_0 is a constant. Eventually [8] it has been realized that both the ballistic, the diffusive and the strong localization regimes should be handled on equal footing using SLRT. The Kubo theory applies in the LRT limit $\varepsilon \to 0$, while in mesoscopic circumstances SLRT leads to a resistor network [13] 'hopping' picture in energy space that generalizes the real space hopping picture of [14, 15].

2. Outline

In this communication we analyze, within the framework of SLRT, the dependence of the mesoscopic conductance of a quasi-1D ring on the strength of the disorder. We explain that for both weak and strong disorder the non-ergodicity of the quantum eigenstates implies having *texture* and *sparsity* in the perturbation matrix. Such features imply that the rate of energy absorption is suppressed enormously because the system cannot execute *connected sequences of transitions*. The implied deviations from the Kubo–Drude result are demonstrated numerically in figure 1.

We introduce a novel random matrix theory (RMT) model, with either *log-box* or *log-normal* distributed elements, that captures the essential features of the perturbation matrix. A generalized resistor network analysis for the EMF-driven transitions in energy space leads to

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[3] Hence there is no issue of quantum recurrences which would arise for a strictly linear or periodic driving. From here on we use units such that $\hbar = 1$.

[4] The terminology of this paper, and in particular our notion of 'conductance' are the same as in the theoretical review [4] and in the experimental work [5].

[5] The term 'semi-linear response' to describe the outcome of the theory of [9] has been coined in a subsequent work [11], where it has been applied to the analysis of the absorption of low-frequency radiation by metallic grains.
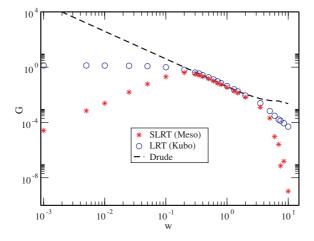
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J. Phys. A: Math. Theor. **41** (2008) 262001   Fast Track Communication

Figure 1. Plot of the scaled conductance $\tilde{G}$ versus W using either the LRT (Kubo) or the SLRT (mesoscopic) recipe, and compared with the Drude-formula-based estimate. The calculation has been carried out for a tight binding Anderson model of size 500×10, transverse hopping amplitude $c = 0.9$ and low driving frequency $\omega_c/\Delta = 7$. The SLRT result departs from the LRT result for both weak disorder (ballistic regime) and strong disorder (strong localization regime).

a generalized VRH picture in the strong disorder limit, while handling on equal footing the opposing limit of very weak disorder.

In the first part of this communication (sections 1–5) we provide the essential details on the model and on the LRT/SLRT calculation, leading to the numerical results in section 6. The second part of this communication leads to the RMT modeling and to the implied generalized VRH picture.

3. Modeling

We consider the disorder quasi-1D ring geometry. The amount of disorder is traditionally characterized by the mean free path ℓ. The semiclassical theory of the conductance leads to the Drude formula[6]

$$G_{\text{Drude}} = \frac{e^2}{2\pi\hbar} \mathcal{M} \frac{\ell}{L}, \tag{2}$$

where L is the length of the ring and $\mathcal{M}$ is the number of open modes (proportional to its cross section). For the numerical calculations we have used the Anderson tight binding model, where the lattice is of size $L \times M$ with $M \ll L$. The longitudinal hopping amplitude per unit time is $c_{\parallel} = 1$, while in the transverse direction it is numerically convenient to have $c_{\perp} < 1$, so as to have in the middle of the band a finite energy window with $\mathcal{M} = M$ open modes.

The random on-site potential in the Anderson tight binding model is given by a box distribution of width W. The density of states at the Fermi energy, and the mean free path in

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[6] Optionally if the ring is characterized by its transmission g, then ℓ/L is replaced by $g/(1-g)$. See [9, 10] for details. As could be expected the result is in agreement with the Landauer theory [16] provided $g \ll 1$. Indeed, the Landauer formula can be obtained from the Kubo formula, using a semiclassical evaluation of the velocity–velocity correlation function, whenever the contribution of trajectories with a non-zero winding number can be neglected.

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J. Phys. A: Math. Theor. **41** (2008) 262001   Fast Track Communication

the Born approximation, are written as

$$\varrho_\mathrm{F} \equiv \mathcal{M} \frac{L}{\pi \hbar v_\mathrm{F}} \equiv \frac{1}{\Delta}; \qquad \ell \sim \left(\frac{v_\mathrm{F}}{W}\right)^2. \tag{3}$$

The implied definition of v_F leads to its identification as the Fermi velocity in the absence of disorder (disregarding a prefactor of order unity). The Anderson localization length for $L = \infty$ sample would be $\ell_\infty = \mathcal{M}\ell$. Accordingly, for a finite sample, depending on the strength of the disorder, one distinguishes between the ballistic regime ($L \ll \ell$), the diffusive regime ($\ell \ll L \ll \ell_\infty$) and the Anderson strong localization regime ($L \gg \ell_\infty$).

4. The LRT calculation

The fluctuation–dissipation version of the Kubo formula expresses the conductance as an integral over the velocity–velocity correlation function:

$$G_\mathrm{LRT} = \varrho_\mathrm{F} \left(\frac{e}{L}\right)^2 \times \frac{1}{2} \int_{-\infty}^{\infty} \langle v(t)v(0)\rangle \, \mathrm{d}t, \tag{4}$$

where v is the velocity in the longitudinal direction. The Drude formula in equation (2) is based on the simplest classical approximation:

$$\langle v(t)v(0)\rangle \approx v_\mathrm{F}^2 \exp\left[-2\left(\frac{v_\mathrm{F}}{\ell}\right)|t|\right]. \tag{5}$$

Our objective is to find the conductance of the closed ring in circumstances such that the motion inside the ring is coherent (quantum interferences are not ignored). The calculation involves the quantum version of $\langle v(t)v(0)\rangle$ which can be obtained as the Fourier transform of the spectral function

$$\tilde{C}(\omega) = \frac{1}{N} \sum_{nm} |v_{nm}|^2 2\pi \delta_\Gamma(\omega - (E_m - E_n)), \tag{6}$$

where N is the size of the energy window of interest. This spectral function can be re-interpreted as describing the band profile of the perturbation matrix $\{v_{nm}\}$. In particular, the calculation of $\tilde{C}(0) \equiv 2\pi \varrho_\mathrm{F} \langle\langle |v_{nm}|^2\rangle\rangle_\mathrm{LRT}$ involves a simple *algebraic* average over the near-diagonal matrix elements at the energy range of interest. Using this notation, the formula for the Kubo conductance takes the form

$$G_\mathrm{LRT} = \pi \left(\frac{e}{L}\right)^2 \varrho_\mathrm{F}^2 \langle\langle |v_{nm}|^2\rangle\rangle_\mathrm{LRT}. \tag{7}$$

The $\mathcal{O}(\Delta/\omega_c)$ weak localization corrections to the Drude formula in equation (2) are determined by the interplay of the broadened delta function in equation (6) with the level statistics. See [4]. Equivalently, we may say that the *algebraic* average $\langle\langle \cdots \rangle\rangle_\mathrm{LRT}$ has some weak sensitivity to the off-diagonal range of the averaging.

5. The SLRT calculation

As in the standard derivation of the Kubo formula, also within the framework of SLRT, the leading mechanism for absorption is assumed to be Fermi-golden-rule (FGR) transitions. These are proportional to the squared matrix elements $|v_{nm}|^2$ of the velocity operator. Still, the theory of [9] does not lead to the Kubo formula. This is because the rate of absorption depends crucially on the possibility to make *connected* sequences of transitions. It is implied that both the texture and the sparsity of the $|v_{nm}|^2$ matrix play a major role in the calculation of

J. Phys. A: Math. Theor. **41** (2008) 262001   Fast Track Communication

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G. SLRT leads to a formula for G that can be cast into the form of equation (7), provided the definition of $\langle\langle\ldots\rangle\rangle$ is modified. Following [11, 10] we regard the energy levels as the nodes of a resistor network. We define

$$g_{nm} = 2\varrho_{\mathrm{F}}^{-3}\frac{|v_{nm}|^2}{(E_n-E_m)^2}\delta_{\Gamma}(E_m-E_n). \tag{8}$$

Then it is argued that $\langle\langle|v_{nm}|^2\rangle\rangle_{\mathrm{SLRT}}$ is the inverse resistivity of the network. It is a simple exercise to verify that if all the matrix elements are the same, say $|v_{nm}|^2 = \sigma^2$, then $\langle\langle|v_{nm}|^2\rangle\rangle_{\mathrm{SLRT}} = \sigma^2$ too. But if the matrix has texture or sparsity then $\langle\langle|v_{nm}|^2\rangle\rangle_{\mathrm{SLRT}} \ll \langle\langle|v_{nm}|^2\rangle\rangle_{\mathrm{LRT}}$.

6. Numerical results

It is natural to define the scaled conductance of the ring as follows:

$$\tilde{G} = \frac{G}{(e^2/2\pi\hbar)\mathcal{M}} = 2\mathcal{M} \times \frac{1}{v_{\mathrm{F}}^2}\langle\langle|v_{nm}|^2\rangle\rangle. \tag{9}$$

This would be the average transmission per channel, if we had considered the open (Landauer) geometry. But for a closed (ring) geometry, $\tilde{G}$ is determined by the appropriate 'averaging' procedure $\langle\langle\ldots\rangle\rangle_{\mathrm{LRT/SLRT}}$. In the SLRT case, the 'averaging' is in fact a resistor network calculation. If all the near-diagonal elements are comparable in size, then SLRT will give essentially the same result as LRT. More generally, $\langle\langle|v_{nm}|^2\rangle\rangle_{\mathrm{SLRT}}$ is typically bounded from above by the *algebraic* average, and bounded from below by the *harmonic* average. The latter is defined as $\langle\langle X\rangle\rangle_{\mathrm{h}} = [\langle 1/X\rangle]^{-1}$, and reflects the 'addition of resistors in series'. If the distribution is not too stretched then the *median*, or the *geometric* average or the *mixed* average of [9] might provide a good approximation. But, in general, a proper resistor network calculation is required. The resistor network calculation is sensitive to the texture and the sparsity of the perturbation matrix. By *texture* we mean that the gray-level image of the v_{nm} matrix appears to be scarred by structures, rather than being homogeneous. Looking on the images of the v_{nm} matrices for various values of W, one realizes that both texture and sparsity emerge in the ballistic case, while in the strong localization case one observes only sparsity. We further expand on the quantitative characterization below.

In figure 1 we plot the Drude conductance $\tilde{G}_{\mathrm{Drude}}$ of equation (2), and the Kubo conductance $\tilde{G}_{\mathrm{LRT}}$, together with the mesoscopic conductance $\tilde{G}_{\mathrm{SLRT}}$ versus W. We see that outside of the diffusive regime, for both weak and strong disorder, the SLRT result is extremely small compared with the LRT expectation. This generic behavior is related to the sparsity and the texture of the perturbation matrix, which is implied by the statistical properties of the eigenstates. The statistical analysis is carried out in figure 2, while the RMT perspective is tested in figure 3. The content of figures 2 and 3 is further discussed in the following sections.

In order to determine numerically whether the *texture* is of any importance we simply permute randomly the elements of the v_{nm} matrix along the diagonals, and re-calculate $\tilde{G}$ (see the 'untextured' data points in figure 3). Obviously, by definition, $\tilde{G}_{\mathrm{LRT}}$ is not affected by this numerical maneuver. But it turns out that $\tilde{G}_{\mathrm{SLRT}}$ is somewhat affected by this procedure in the ballistic regime, but still the qualitative results come out the same. Accordingly, we deduce that the main issue is the *sparsity*, and concentrate below on the RMT modeling of this feature.

In the remainder of this paper, we pave an analytical approach to the calculation of the conductance, which will allow us to shed some light on these numerical findings. First, we discuss the familiar diffusive regime where both LRT and SLRT should be in agreement with
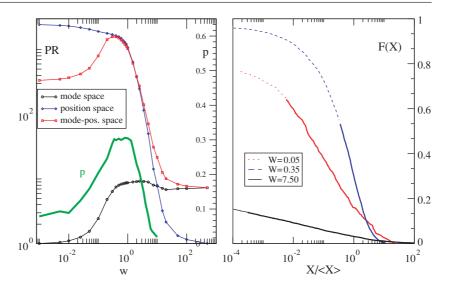
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J. Phys. A: Math. Theor. **41** (2008) 262001   Fast Track Communication

Figure 2. The ergodicity of the eigenstates is characterized by the participation ratio $PR \equiv [\sum \rho^2]^{-1}$, which is calculated (left panel) in various representations: in position space $\rho_{r_x,r_y} = |\langle r_x, r_y|\Psi\rangle|^2$, in position-mode space $\rho_{r_x,k_y} = |\langle r_x, k_y|\Psi\rangle|^2$, and in mode space $\rho_{k_y} = \sum_{r_x} |\langle r_x, k_y|\Psi\rangle|^2$, where $k_y = [\pi/(\mathcal{M}+1)] \times$ integer. The cumulative distribution $F(X)$ of the in-band matrix elements (right panel) exhibits a log-box distribution in the strong localization regime. Points in the interval $X > \langle\langle X \rangle\rangle_{SLRT}$, corresponding to non-negligible values, are connected by a thicker line. The extracted sparsity measure (left panel) is $p \equiv F(\langle X \rangle)$.
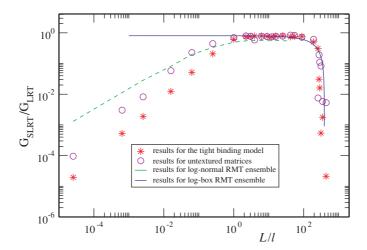

Figure 3. The ratio G_{SLRT}/G_{LRT} versus the inverse of $\tilde{G}_{Drude} = \ell/L$ based on the numerics of figure 1, and compared with artificial RMT modeling using 'sparse' matrices formed of log-normal or log-box distributed elements. We also compare the actual results with 'untextured' results as explained in the text. For weak disorder the agreement is only qualitative indicating that the texture becomes important.

the Drude approximation (the latter should become a good approximation for a big sample). Then we discuss the departure of SLRT from LRT outside of the diffusive regime, which reflects the sparsity of the v_{nm} matrix due to the non-ergodicity of the eigenfunctions.

J. Phys. A: Math. Theor. **41** (2008) 262001   Fast Track Communication

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7. The random wave conjecture

In the diffusive regime, Mott has argued that the eigenstates of the Hamiltonian matrix are ergodic in position space, and look like random waves. Using this assumption one can reconstruct the Drude result. Following Mott[7] we assume that ℓ is the correlation scale of any typical eigenfunction $\Psi(x, y)$. The basic assumption of Mott is that the eigenstates are locally similar to free waves. The total volume L^d is divided into domains of size ℓ^d. Hence we have $(L/\ell)^d$ such domains. Given a domain, the condition to have non-vanishing overlap upon integration is $|\vec{q}_n - \vec{q}_m|\ell < 2\pi$, where $\vec{q}$ is the local wave number within this domain. The probability that $\vec{q}_n$ would coincide with $\vec{q}_m$ is $1/(k_E\ell)^{d-1}$. The contributions of the non-zero overlaps add with random signs hence

$$|v_{nm}| = \left[\frac{1}{(k_E\ell)^{d-1}} \times \left(\frac{L}{\ell}\right)^d \right]^{1/2} \times (\overline{\Psi^2}\ell^d)v_F, \tag{10}$$

where the assuming ergodicity $\overline{\Psi^2} \approx 1/L^d$. From here we get $\tilde{G} \sim \ell/L$, leading to the Drude result. We discuss the limited validity of this result in the following section.

8. The non-ergodicity issue

It is clear that Mott's derivation of the Drude formula on the basis of LRT and the random wave conjecture becomes non-applicable if the eigenfunctions are non-ergodic. This is indeed the case for both weak and strong disorder: a typical eigenfunction does not fill the whole accessible phase space. In the ballistic regime a typical eigenfunction is not ergodic over the open modes in momentum space, while in the strong localization regime it is not ergodic over the ring in real space [17]. Figure 2 demonstrates this point by plotting the participation ratio as a function of the disorder strength. Lack of quantum ergodicity for either weak or strong disorder implies that the perturbation matrix v_{nm} is very textured and/or sparse. For the following analysis, a precise mathematical definition of sparsity is required. In the following sections we shall provide such a definition, but for this purpose we have to shed some light on the size distribution of the matrix elements.

In the strong disorder regime, the observed 'sparsity' is very simple for understanding: eigenstates that are close in energy are typically distant in real space, and therefore have very small overlap. The 'big' matrix elements are contributed by eigenstates that dwell in the same region in real space, and hence sparse in energy space. What we are going to call in the following sections 'sparsity', is merely a reflection of the associated log-box size distribution of the matrix elements (see figure 2(*b*)). The log-box distribution is deduced by a straightforward extension of the above argument. A generic eigenfunction in the localized regime has an exponential shape $\psi(r) \sim \exp(-|r - r_0|/\ell_\infty)$, which is characterized by the localization length ℓ_∞. Consequently, a typical matrix element of $\{|v_{nm}|^2\}$ has the magnitude

$$X \sim \frac{1}{\mathcal{M}^2} v_F^2 \exp\left(-\frac{x}{\ell_\infty}\right), \tag{11}$$

where $x \in [0, L/2]$ has a uniform distribution. The prefactor is most easily derived from the requirement of having $\langle X \rangle \approx (\ell/\mathcal{M}L)v_F^2$ in agreement with the semiclassical result. The latter is deduced from the Fourier transform of the velocity–velocity correlation function (5).

In the weak disorder regime, the explanation of the observed 'sparsity' and textures requires some more effort. For the purpose of this communication we shall be satisfied with

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[7] The original argument by Mott is somewhat vague. We thank Holger Schanz for helpful communication concerning a crucial step in the derivation.

a qualitative explanation: if the disorder W were zero, then the mode index (call it n_y) would become a good quantum number. This means that states that are close in energy are not coupled (because they have different n_y). Once W becomes non-zero (but still small) the mixing is described by Wigner Lorentzians (much the same as in the toy model of [9]). Then the ratio between small and large couplings is determined by the different degree of mixing of close versus far modes. Consequently, one observes a wide (but not stretched) distribution for the $\log(X)$ values (see figure 2(b)).

9. RMT modeling, beyond the Gaussian assumption

It was the idea of Wigner [18] to model the perturbation matrix of a *complex* system as a (banded) random matrix. Later it has been conjectured by Bohigas [19] that similar modeling may apply to *chaotic* systems. For many purposes it is convenient to assume infinite bandwidth with elements that are taken from a Gaussian distribution, leading to the standard Gaussian Orthogonal or Unitary ensembles (GOE/GUE). But there are obviously physical circumstances in which it is essential to go beyond the Gaussian assumption, and to take into account the implications of having finite bandwidth and/or non-Gaussian distribution of elements [20] and/or sparsity [21] and/or texture.

It should be clear that the default assumption of having a Gaussian distribution of in-band matrix elements is legitimate on practical grounds as long as the matrix elements have *comparable size in absolute value*. But if the eigenfunctions are non-ergodic this assumption becomes problematic, because the elements (in absolute value) might have a wide distribution over many decades in the log scale. In such a case different type of averages may differ by orders of magnitude.

In the following, we regard $\{|v_{nm}|^2\}$ as a random matrix of non-negative numbers $\{X\}$. In general, it might be a banded matrix. If the standard Gaussian assumption applies, then the in-band elements of $\{X\}$ are characterized by the Porter–Thomas distribution. But we are interested in physical circumstances in which many of the in-band elements are vanishingly small. We define this feature as 'sparsity'. In the following section we define p as the fraction of elements that are larger than the average. If we have $p \ll 1$ then we say that the matrix is 'sparse'. We further discuss the definition of p in the following section.

10. Characterization of sparsity

For an artificially generated sparse random matrix $\{X\}$ of non-negative elements, one defines p as the fraction of non-zero elements. Such a definition assumes a bimodal distribution. But in general realistic circumstances we do not have a bimodal distribution. Rather for strong disorder we already had explained that the distribution of the matrix elements $\{|v_{nm}|^2\}$ is log-box. Contemplating a bit on this issue one concludes that the physically generalized definition of the sparsity measure is $p \equiv F(\langle X \rangle)$, where $F(X)$ is the probability to find a value larger than X. We regard a matrix as sparse if $p \ll 1$. Given that $\ln(X)$ is uniformly distributed between $\ln(X_0)$ and $\ln(X_1)$ we define $\tilde{p} \equiv (\ln(X_1/X_0))^{-1}$, and find assuming $X_0 \ll X_1$ that $p \approx -\tilde{p} \ln \tilde{p}$, and $\langle X \rangle \approx \tilde{p} X_1$. Hence for log-box distribution $\langle X \rangle \sim p X_1$, as expected from the standard bimodal case.

In figure 3, we redo the calculation of the conductance with artificial matrices with the same sparsity, i.e. log-box distributed elements with the same p. We observe qualitative agreement for strong disorder. In the other extreme limit of weak disorder there is no agreement, because we have to use a different distribution for the matrix elements: it turns out that also in the

ballistic regime $\log(X)$ has a wide distribution, but it is not stretched as in the case of a log-box distribution (see section 8). In practice, we can describe the X distribution of the matrix elements in the ballistic limit as log-normal[8]. Once we use the appropriate distribution we get a reasonable qualitative agreement. We emphasize that in both cases, of either weak or strong disorder, there is besides the algebraic average $\langle X \rangle$ only *one* additional fitting parameter that characterizes the distribution and hence determines the 'sparsity'. We could of course have generated RMT matrices using the actual distribution (figure 2), but then we would merely re-generate the *untextured* data points.

Given a hopping range $|E_m - E_n| \leqslant \omega$ we can look for the typical matrix element $\overline{X}$ for connected sequences of such transitions, which we find by solving the equation

$$\left(\frac{\omega}{\Delta}\right) F\left(\overline{X}\right) \sim 1. \tag{12}$$

In particular, for strong disorder we get

$$\overline{X} \approx v_F^2 \exp\left(-\frac{\Delta_\ell}{\omega}\right), \tag{13}$$

where $\Delta_\ell = (L/\ell_\infty)\Delta$ is the local level spacing between eigenstates that are localized in the same region. The same procedure can be applied also in the ballistic regime leading to a simpler variation of (13) where the dependence of $\overline{X}$ on ω predominantly reflects the band profile: it follows from the discussion after equation (6) that v_{nm} is a banded matrix, with a Lorentzian band profile whose width $\sim v_F/\ell$ becomes narrower as the disorder is decreased.

11. Generalized Kubo formula

The definition of the band profile reflects the variation of $\langle X \rangle$ with ω. In complete analogy, we define an effective band profile that reflects the variation of $\overline{X}$ with ω. Namely,

$$\tilde{C}_{\text{qm-LRT}}(\omega) \equiv 2\pi\varrho_F\langle X \rangle, \tag{14}$$

$$\tilde{C}_{\text{qm-SLRT}}(\omega) \equiv 2\pi\varrho_F\overline{X}. \tag{15}$$

The spectral function of equation (6) is a smeared version of the 'bare' spectral function: it is obtained by a convolution $\tilde{C}_{\text{qm-LRT}}(\omega) \star \delta_\Gamma(\omega)$. Consequently, we get

$$G = \frac{1}{2}\left(\frac{e}{L}\right)^2 \varrho_F \int \tilde{C}_{\text{qm}}(\omega)\delta_\Gamma(\omega)\,\mathrm{d}\omega, \tag{16}$$

where the appropriate LRT/SLRT spectral function should be used. This way of writing allows us to obtain an approximation for the mesoscopic conductance using a Kubo-like calculation: it is just re-writing of the Kubo formula in the LRT case, while being a generalized VRH approximation in the SLRT case.

For strong disorder the above generalized VRH approximation gives an integral over $\exp(-|\omega|/\omega_c)\exp(-\Delta_\ell/|\omega|)$, which is a product of two competing factors: the first has to do with the noise/temperature and the second has to do with the couplings. This integral is handled using the usual VRH phenomenology: the result is determined by the maximum of its integrand, which requires to optimize the range ω of the transition. In the weak disorder regime the VRH integral is not the same as in the strong disorder case, because a log-normal

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[8] The default fitting of the $\log(X)$ distribution to a *Gaussian* line shape is merely a practical issue. The 'RMT ideology' is to see whether a 'minimum information' ensemble of random matrices can be used in order to derive reasonable estimates. If we want to further improve our estimates in the ballistic regime it is essential to take into account the texture and not only the deviation from log-normal distribution (see section 6).

J. Phys. A: Math. Theor. **41** (2008) 262001

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rather than log-box distribution is involved. We have verified that the generalized VRH integral gives a qualitatively reasonable approximation to the actual resistor network calculation in both cases. In any case one should keep in mind the well-known reservations that apply to such 'mean field' approach [14, 15].

12. Summary

Within SLRT it is assumed that the transitions between levels are given by the Fermi-golden-rule (FGR), but a resistor network analogy is used in order to calculate the energy absorption rate. The calculation generalizes the variable range hopping picture and treats on equal footing the weak and strong disorder regimes. The essential physics is captured by RMT, provided the perturbation matrix is regarded as a member of the appropriate Gaussian/log-normal/log-box ensemble.

The prevailing results in the literature regarding the conductance of small closed metallic rings (for a review, see [4]) concern mainly the diffusive regime, where in leading order the conductance is given by the Drude formula, and SLRT does not differ much from LRT. In the present communication, multi-mode rings in the non-diffusive regime are considered seriously for the first time. Then it become essential to define the precise assumptions regarding the environment and the driving. It is important to realize that both LRT and SLRT assume Markovian FGR transitions. This is a very realistic assumption that can be justified rigorously if one assumes noisy driving (as in our exposition) or else it is implied by having a noisy environment[9]. Accordingly, it should be clear that LRT and SLRT both share the same small parameter as in the FGR picture, which is the ratio between the rate of the driven transitions and the smallest relevant energy scale that characterizes the band profile (bandwidth/sparsity/texture).

There is only one assumption that distinguishes the SLRT (mesoscopic) circumstances from LRT (Kubo) circumstances. This is related to the implicit role of the environmentally induced relaxation process in the determination of the steady state of the system. Within SLRT one assumes that the FGR rate of the driven transitions ($w_{\text{FGR}} \propto \varepsilon^2 \mathsf{g}_{nm}$) is larger compared with the relaxation rate (γ_{rlx}). The inelastic relaxation effect can be incorporated into the SLRT framework by considering a non-symmetric g_{nm} as implied (say) by detailed balance considerations. If the relaxation is the predominant effect ($w_{\text{FGR}} < \gamma$), then we are back in the LRT regime [23] where the Kubo–Drude result applies [9].

One can wonder what happens if the FGR assumption of LRT/SLRT breaks down. Not much is known [24]. Reference [25] has attempted to go beyond the FGR approximation using the Keldish formalism, and has recovered a Markovian picture that leads to a Kubo-like result for the conductance. If the Keldish Markovian picture could be established beyond the diffusive regime [26], it would be possible to extend SLRT into the nonlinear regime.

[9] The possibility to witness dynamical localization-related corrections [22] that go beyond the FGR picture requires strictly coherent microscopic-like circumstances, such that the dephasing time is much longer compared with the *Heisenberg time*, and the low-frequency driving is required to be strictly periodic over those extremely long periods. Such conditions are possibly not easy to achieve in realistic experimental circumstances once multi-mode rings are concerned.

J. Phys. A: Math. Theor. **41** (2008) 262001

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Fast Track Communication

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