Master Thesis

The Axion-Instanton Weak Gravity Conjecture and Scalar Fields

Submitted by:
Clemens Vittmann

Supervised by:
PD Dr. Eran Palti

Referees:
Prof. Dr. Arthur Hebecker
Prof. Dr. Timo Weigand

This thesis has been carried out at the
Institute for Theoretical Physics

Saturday 4th August, 2018
Foremost, I thank Eran Palti for giving me the opportunity to write this thesis, for guiding me through the process and for the hours of discussing and explaining even after he left Heidelberg University for a position in Munich.

Second, I thank Arthur Hebecker, whose lectures introduced me to quantum field theory, and Timo Weigand, who kept me busy with string theory at the same time, for being willing to be the referees of my thesis.

I am grateful to Marc Merstorf and Clemens Fruböse as well as Sascha Leonhardt and Thomas Mikhail for tons of helpful and entertaining discussions (not only) about physics and for providing a very pleasant work atmosphere at the Philosophenweg 19.

My last thanks goes to Mikio Nakahara for writing his excellent textbook on differential geometry, topology and physics which taught me most of the mathematics necessary for dealing with complex manifolds.
We study the Weak Gravity Conjecture in the presence of scalar fields. The Weak Gravity Conjecture is a consistency condition for a theory of quantum gravity asserting that for a $U(1)$ gauge field, there is a particle charged under this field whose mass is bounded by its charge. It was extended to a statement about any canonical pair of $(p−1)$-dimensional object and $p$-form coupling to it, in particular to axion-instanton pairs. The gauge-scalar Weak Gravity Conjecture is a modification of this bound that includes scalar interactions. We propose a similar extension to cases where scalar fields are present for the axion-instanton Weak Gravity Conjecture and provide evidence from Type IIA supergravity.

## Contents

1 Introduction .................................................. 1

2 Weak Gravity Conjecture .................................... 7
   2.1 Absence of global symmetries in quantum gravity ........... 7
   2.2 Weak Gravity Conjecture .................................. 8
   2.3 Generalized Conjectures .................................. 11

3 Calabi-Yau compactification ................................. 21
   3.1 Kaluza-Klein reduction .................................. 21
   3.2 Calabi-Yau requirement .................................. 23
   3.3 Reduction on the Calabi-Yau ............................. 26
   3.4 Moduli spaces of Calabi-Yau threefolds ................... 27
   3.5 Complex structure moduli space .......................... 30
   3.6 Kähler moduli space .................................... 33
   3.7 Mirror symmetry ...................................... 35

4 A Scalar WGC for Type IIB Particles ....................... 37
   4.1 Supersymmetric black holes .............................. 37
   4.2 Gauge fields from reduction of Type IIB supergravity .... 41
   4.3 Particles from D3-branes ............................... 46
   4.4 A scalar WGC for the particle ......................... 48

5 A Scalar WGC for Type IIA Instantons .................... 53
   5.1 Axions from reduction of Type IIA supergravity .......... 53
   5.2 Instantons from E2-Branes .............................. 56
   5.3 A scalar WGC for instantons and axions ................ 57

6 Conclusion ..................................................... 61

A Mathematical Preliminaries ................................. 63
   A.1 Complex manifolds ...................................... 63
   A.2 Kähler geometry ...................................... 67
   A.3 Calabi-Yau manifolds .................................. 69
   A.4 Some integrals on Calabi-Yau threefolds ................. 71

B Type II Supergravity .......................................... 73
   B.1 Type IIA SUGRA from compactification of 11-dim. SUGRA .... 73
   B.2 Type IIB SUGRA ....................................... 75
   B.3 $\mathcal{N} = 2$ supergravity in $d = 4$ and special geometry ... 76
C Calculations
    C.1 Compactification 81
    C.2 Supersymmetric black holes 83
    C.3 Gauge-coupling matrix 83

References 93
List of Figures

1 Feynman diagram and string worldsheet ....................... 2
2 Swampland surrounding string landscape ..................... 3
3 Particles satisfying gauge-salar WGC .......................... 4
4 D3-brane wrapping three-cycle ................................ 5
5 Increasing black hole charge ................................ 7
6 Energy conservation for decaying black hole ................. 8
7 Extremal black hole decaying ................................ 11
8 Convex hull condition ....................................... 14
9 Moduli space of a quantum theory of gravity ................. 18
10 Compactification of two-dimensional surface ................. 21
11 Running couplings in minimal supersymmetric extension of standard model ........................................ 23
12 Kähler cone ............................................... 28
13 E2-brane wrapping three-cycle ............................... 56
14 Hodge Diamond of Calabi-Yau threefold .................... 70

All figures were created by the author.

List of Tables

1 Bases for cohomologies ....................................... 27
2 Metrics and Kähler potentials ............................... 36
3 Type IIB multiplets .............................................. 43
4 Type IIA multiplets .............................................. 54
5 $\mathcal{N} = 2$ supergravity multiplets ......................... 77
1 Introduction

When attempts to find a description of the strong interaction were made fifty years ago - the force binding protons and neutrons together - young Gabriele Veneziano found that the Euler beta function had certain features one would expect from the scattering amplitude of strongly interacting particles. No theory was known at that time, though, that would produce such a scattering amplitude. It was about two years later that Yoichiro Nambu, Holger Bech Nielsen and Leonard Susskind independently discovered that it was not a theory of point particles but one of vibrating strings that would give rise to Veneziano’s amplitude. Soon, those working on the physics of such strings realized two things: First, this model was making predictions about the strong force which were not in accordance with the experimental findings and the theory of quarks and gluons developed at that time - quantum chromodynamics - turned out to be superior as a description of the strong interaction. Second - and quite surprisingly - this theory of strings seemed to offer a solution to a very different but fundamental problem of theoretical physics: Bringing together Einstein’s general relativity and the Standard Model in a single unifying theory.

After the two revolutions of theoretical physics in the first half of the twentieth century - the discoveries of general relativity and quantum mechanics - these two fields developed essentially separate from each other and appeared to be drastically different: For example, in quantum field theory - the relativistic quantum framework now underlying our description of the electromagnetic, weak and strong interactions in what is known as the Standard Model - fields at two points in space-time whose separation is space-like should (anti-)commute. Meanwhile, in general relativity - the theory describing the remaining fundamental interaction, namely gravity - the metric is dynamical and one does a priori not even know whether a distance is space-like [1]. Above all, general relativity is classical and one runs into the following difficulties when attempting to perform the same procedure of perturbative quantization which proved so successful for the other fundamental interactions: Unlike the Standard Model, gravity is non-renormalizable meaning that the infinities one encounters in the quantization process cannot be removed by a finite number of counter-terms. Although the resulting theory is predictive at low energies - i.e. below the Planck mass - it breaks down at shorter distances [2]. While this does not play a role for most practical purposes in physics, things do become problematic in the quantum description of small black holes and early universe cosmology.

Without elaborating on the idea of unification as a driving force in the history of physics, we want to briefly review why it is plausible to assume that the Standard Model and gravity themselves are - in spite of their success in describing basically all experimentally observed phenomena - not fundamental descriptions but rather
limits of such a complete theory. We already mentioned the problem of infinities in quantum field theories. While it is well-known that the Standard Model is a renormalizable QFT, which means that these infinities can be absorbed in a finite number of counter terms, their very appearance suggests that this description is merely an effective one, a low-energy limit of a more fundamental theory. The same is true with regard to the spacetime singularities in general relativity which are contained in the centers of black holes. The circumstance that the Standard Model has about twenty free parameters - like electron mass or mixing angles which have to be taken from experiment - renders the theory arbitrary. Even worse, some of these parameters appear to be fine-tuned. We will not elaborate on this any further but go back to the UV infinities of quantum field theory. It was

realized that if the fundamental entities of nature were not point-like but rather of finite length, then these UV divergences would not occur. This was proved for the one- and two-loop diagrams analogous to the Feynman diagrams of point particles and there is no reason to expect anything different at higher orders. Even without performing the actual calculation, this behavior is intuitively accessible when looking at such stringy diagrams like the one in fig. 1 on the right-hand side: Unlike the case of point particles, there is no single point in spacetime where scattering strings interact. Instead, the worldsheet in fig. 1 always looks locally like that of a single freely propagating string and it is only the topology of this worldsheet which encodes any interaction.

Even more surprisingly, the theory included a particle in its spectrum that had precisely the numbers of freedom one would expect from a particle that carries the gravitational force, the graviton. At the same time, some problems like the ambiguity arising from free parameters would not appear: String theory has no free dimensionless parameters, all couplings are expectation values of fields and

---

**Figure 1:** Feynman diagram of point particles (l.h.s.) and closed strings scattering diagram (r.h.s.).
the theory seems to be an up to dualities unique and consistent theory of quantum gravity.

While string theory is unique in ten spacetime dimensions, there are many consistent ways to obtain a four-dimensional effective theory corresponding to the choice of compactification manifold. The set of these solutions to string theory is known as the string landscape and one is led to the question if, conversely, any consistent effective field theory can be coupled to gravity\(^1\). About thirteen years ago, Vafa argued [3] that this was not the case and termed the set of those theories which lack quantum consistency the swampland.

Several criteria were proposed [3, 4] to distinguish effective theories that can be UV-completed to a consistent theory of quantum gravity from those that merely lie in the “surrounding” swampland. The arguably best known criterion to exclude a theory from the landscape is the “folk theorem” that quantum gravity does not allow for global symmetries. This is turned into a quantitative statement by the conjecture proposed by Arkani-Hamed et al. in [5] that “gravity must be the weakest force”. More precisely: For a consistent quantum theory containing gravity and a \(U(1)\) gauge field there must exist a particle charged under this gauge field with a mass bounded from above by its charge. This Weak Gravity Conjecture (WGC) has attracted much interest in the past decade and found application e.g. in cosmology, where it is used to constrain models of large field inflation [6, 7, 8]. It was generalized to settings with several \(U(1)\)s [9], \(p\)-forms [10] and quite recently to situations where scalar fields are present [11, 12, 13].

This last generalization is particularly important for us. It proposes bounds (henceforth called Scalar and Gauge-Scalar WGC)

\[
m^2 < \mu^2 M_p^2, \quad g^2 M_p^2 \geq m^2 + \mu^2 M_p^2
\]  

\(^1\)If that were the case, one would not try to construct four-dimensional theories as compactifications of ten-dimensional string theory but rather take some effective theory that fits best to experiment [3].
for a particle \( m \) coupled to a scalar field with coupling \( \mu \). The second inequality is saturated by BPS states and for two such particles, the combined scalar, vector and gravitational forces cancel as sketched in fig. 3. We will elaborate on this in section 2.3.4. In the main part of this thesis, we will eventually suggest a similar condition for \textit{axion-instanton} pairs in the presence of scalar fields. Before doing so, we need to lay the basis. Section 2 begins with a review of the no-global-symmetries conjecture and the black hole based argument supporting it. The remaining part of the section contains a review of the Weak Gravity Conjecture, a discussion of motivation as well as evidence for it and we will present some of the extensions to more general settings.

Evidence for our conjecture about axions, instantons and scalar fields will come from compactification of Type IIA string theory where the scalars will be the moduli of the compactification manifold. In order to lay the ground for sections 4 and 5 and make the thesis self-contained - we review the idea and procedure of Calabi-Yau compactification in section 3, focusing on the moduli spaces of Calabi-Yau manifolds. In particular, we discuss the \textit{special Kähler} structure, which the complex structure and Kähler moduli spaces carry. While a basic knowledge of differential geometry is of course indispensable, appendix A gives a recap of complex manifolds while B.3.2 introduces the notion of \textit{special geometry}.

In section 4 which marks the beginning of the main part of this thesis, we will see that compactification of Type IIB supergravity on a Calabi-Yau three-fold gives rise to several vector fields in spacetime while D3-branes wrapping supersymmetric three-cycles look like particles charged under these gauge fields from the perspective of the four-dimensional theory. The latter is due to the fact that we can perform the spatial integrations in the brane action which results in a one-dimensional path in spacetime as illustrated in fig. 4. We will show that these particles satisfy the condition

\[
q_A G^{AB} q_B = 8 \mathcal{V} \left( m^2 + 4 G^{AB} \nabla_A m \nabla_B m \right).
\] (1.2)
Here, $q_A$ are the charges corresponding to the vector fields, $G^{AB}$ is the metric on the moduli space, $V$ the compactification manifold’s volume, the particle mass $m$ is a function of the moduli and units are chosen such that $M_p = 1$.

The same can be done in Type IIA supergravity, where Euclidean E2-branes couple to the axions $\xi$ and $\tilde{\xi}$ arising from one of the forms upon compactification to four dimensions. In section 5, we will find that wrapping supersymmetric three-cycles, these E2-branes look like points in space-time and can therefore be interpreted as instantons. We will show that these satisfy a condition which is similar to the Gauge-Scalar WGC and relates instanton action, axion decay constants and the scalars:

$$Q^2 = S^2 + G^{ab} \nabla_a S \nabla_b S$$  \hspace{1cm} (1.3)

with $S$ the instanton action and $Q^2$ defined as

$$Q^2 = \frac{1}{2} (p \quad q) \begin{pmatrix} -(I + RI^{-1}R) & R I^{-1} \\ I^{-1} R & -I^{-1} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$ \hspace{1cm} (1.4)

In this expression, the charges $q$ and $p$ are the couplings to $\xi$ and $\tilde{\xi}$ respectively while the matrix entries come from the kinetic terms for the axions in the four-dimensional action. The matrices $I$ and $R$ are the imaginary and real parts of the gauge-coupling matrix $\mathcal{M}$ which is defined in terms of the prepotential for the
Calabi-Yau moduli. We will establish (1.3) for the axions with kinetic term

$$\int \left[ (\mathcal{I})^{-1} \right]^{\hat{a} \hat{b}} d\tilde{\xi}^a \wedge *d\tilde{\xi}^b$$  \hspace{1cm} (1.5)

which corresponds to having purely $p$-charges but expect it to hold also for charges $q$ or both, $q$ and $p$.

The Gauge-Scalar WGC can be phrased as the statement that for two particles, the gauge repulsion exceeds the combined gravitational and scalar attraction while all forces cancel in case of two BPS states. It is therefore natural to propose that analogously, the equality (1.3) becomes an inequality

$$Q^2 \geq S^2 + G^{ab} \nabla_a S \nabla_b S$$  \hspace{1cm} (1.6)

in the absence of supersymmetry. For a single axion with decay constant $f$ and only one scalar field $\phi$, this translates to

$$S^2 + \partial_b S \bar{\partial}_b S \leq 1/f^2$$  \hspace{1cm} (1.7)

Further, we conjecture that this relation is a general extension of the axion-instanton WGC to situations with scalar fields.

Finally, some (especially lengthy) calculations were put in appendix C and are cited when needed in order to make the thesis clear and readable.
2 Weak Gravity Conjecture

2.1 Absence of global symmetries in quantum gravity

Since in the limit $g \to 0$ a gauge symmetry becomes a global one, the WGC is strongly motivated by the statement that quantum gravity does not allow for global symmetries. Before giving a sharper formulation of the conjecture, we review the general black-hole based argument [14, 15] for the absence of global symmetries. While a black hole charged under a gauge theory needs to satisfy the extremality bound

$$M \geq QM_p, \quad (2.1)$$

for a global symmetry we can construct a black hole at a fixed mass with arbitrarily high charge $Q$: We can always increase the charge by throwing enough charged particles into the black hole while keeping the black-hole mass constant. The latter is achieved by waiting for the excess mass to be radiated away by emission of uncharged particles (e.g. photons) via Hawking radiation before throwing another particle into the black hole. This is depicted in fig. 5.

![Figure 5: Throwing particles into the black hole to increase its charge to $Q$ while radiating the excess mass away.](image)

Having established that we can consider a black hole with any global charge $Q$, we now see what happens if we let such a black hole evaporate its mass. At some point, its Hawking temperature $T_H = M_p^2/M$ will eventually exceed the mass $m$ of the lightest charged particle. In order to get rid of its charge, the black hole needs to have a mass that is equal to at least $Q$ times $m$ at this point,

$$M \geq Qm. \quad (2.2)$$

---

2) See equation (2.14) below.
This situation is sketched in fig. 6 on the right-hand side. From the two bounds (2.1) and (2.2) we conclude

\[ Q \leq \left( \frac{M_p}{m} \right)^2, \quad (2.3) \]

which is violated if we only take \( Q \) big enough.

While a similar argument can also be applied for gauge theories with tiny couplings [5], in the case of a global symmetry, Hawking radiation produces equal numbers of particles with charge \(+q\) and \(-q\) since there is no electric field outside the black hole to produce a chemical potential term that favors discharge [14]. Thus, the black hole cannot decay completely and what remains is a stable black hole remnant, an object with size and mass of Planck order. Since there is no upper bound on the global charge, this leads to an infinite number of such remnants, which are argued [16] to let the entropy per unit area go to infinity and Newton’s constant to zero. This strongly suggests that global symmetries should be absent from a quantum theory of gravity which is further supported by string theory, where in fact all symmetries are gauged [17].

### 2.2 Weak Gravity Conjecture

In view of the above claim, something should prevent us from taking the limit \( g \to 0 \) for the coupling of a local symmetry where it becomes indistinguishable from a global one. Naturally, the question arises if small but non-zero couplings are also problematic and if so, what the lower bound for allowed couplings is. In their paper [5], Arkani-Hamed et al. proposed the following:
2.2 Weak Gravity Conjecture

In a theory containing a $U(1)$ gauge field with coupling $g$ and gravity,

i) there must be a particle carrying charge $Q$ under the gauge field with its mass satisfying the bound

$$m \leq qM_p,$$ 

(2.4)

where we defined $q = \sqrt{2}Qg$,

ii) and the effective theory breaks down at a scale $\Lambda \lesssim gM_p$.

These statements are called electric and magnetic WGC. The latter actually follows from the former by considering a magnetic monopole: The Dirac-quantization condition demands $g_{\text{mag}} \sim 1/g_{\text{el}}$. Since the monopole mass $m$ needs to account for the energy stored in its magnetic field, we have

$$m \geq g_{\text{mag}}^2 \Lambda,$$ 

(2.5)

where $\Lambda$ is the cutoff of the effective theory. Applying the electric WGC,

$$m \lesssim g_{\text{mag}} M_p,$$ 

(2.6)

it follows that

$$g_{\text{mag}} \Lambda \lesssim M_p,$$ 

(2.7)

which is the magnetic WGC for $g_{\text{mag}} \sim 1/g_{\text{el}}$. Phrased differently, the WGC thus suggests that there is a lower bound on the strengths of interactions associated with the gauge boson and hence turns the merely qualitative argument that global symmetries should be absent from a quantum theory of gravity into a quantitative criterion. Before discussing the conjecture in more detail, we want to point out that while generally thought to be true, the WGC still is somewhat speculative and the best evidence so far is that all models obtained from string theory seem to satisfy it [18]. We will test the conjecture (or rather an extension of it) explicitly in section 4.

The argument we made to exclude global symmetries relied on the fact that the charge of the black hole was not observable from outside, which is no longer true if we consider non-zero gauge couplings. Nevertheless, we can adapt the above argument as follows: Consider a black hole in four dimensions with mass $M$ electrically charged under a $U(1)$ field with coupling $g$. Such a solution of the
Einstein action
\[ S = \frac{1}{2\kappa^2} \int R \ast 1 - \frac{1}{2g^2} \int F \wedge *F \] (2.8)
is called a \textit{Reissner-Nordström black hole}. We chose convention such that
\[ F = dA, \quad A = \frac{g^2Q}{4\pi r}, \] (2.9)
i.e. the charges are defined as
\[ Q = \frac{1}{g^2} \int_{S^2} *F. \] (2.10)
Its metric is [19]
\[ ds^2 = -\Delta dt^2 + \Delta^{-1}dr^2 + r^2d\Omega_2^2, \] (2.11)
where
\[ \Delta = 1 - \frac{2MG}{r} + \frac{(gQ)^2G}{4\pi r^2} \]
\[ = 1 - \frac{M\kappa^2}{4\pi r} + \frac{(gQ)^2\kappa^2}{4\pi(8\pi)r^2} \] (2.12)
with \( \kappa = M_p^{-1} \). Looking for the roots of \( \Delta(r) \),
\[ r = MG \pm \sqrt{\left(\frac{M\kappa^2}{8\pi}\right)^2 - \frac{g^2Q^2\kappa^2}{(4\pi)(8\pi)}}, \] (2.13)
we can see that the \( r = 0 \) singularity is only shielded by an event horizon if the \textit{extremality bound}
\[ M \geq \sqrt{2gQM_p} \] (2.14)
is satisfied. According to the cosmic censorship conjecture [19, 20], naked singularities should not appear in physical situations and therefore, (2.14) needs to be fulfilled. We demand that an extremal black hole is still able to decay. If we label the final states by an index \( i \) and write \( m_i, q_i \) for their masses and charges respectively as depicted in fig. 7, energy conservation demands that the black hole
mass $M = \sqrt{2gQ}M_p$ amounts to at least the sum of the masses of these states,

$$M \geq \sum_i m_i. \quad (2.15)$$

At the same time, we have

$$Q = \sum_i q_i \quad (2.16)$$

due to charge conservation. Until now, we did not give any specification regarding the WGC particle. We will now argue that the weak gravity bound is satisfied by the particle whose charge-to-mass ratio is maximal. Let $z_i := q_i/m_i$. Then

$$Q = \sum_i z_i m_i \leq z_{\text{max}} \sum_i M = z_{\text{max}} M, \quad (2.17)$$

where $z_{\text{max}} = q/m$ denotes the particle with maximal charge-to-mass ratio. Using $M = \sqrt{2gQ}M_p$, we conclude

$$1 \leq \sqrt{\frac{g}{m}} q M_p, \quad (2.18)$$

which is precisely the electric Weak Gravity Conjecture (2.4).

### 2.3 Generalized Conjectures

If we take a look at the weak gravity bound (2.4) again, we find that there are different possible ways in which it could be modified. First, one can ask if an analogous bound holds also for higher-dimensional objects charged under some $p$-form. Indeed, the argument we just gave is not restricted to zero-dimensional
objects, i.e. particles. Second, we can stick to particles but consider a setup where these are charged under several gauge fields and see how this restricts the particle mass. At last, we can take additional - namely scalar - forces into account. We will now discuss these modifications in turn.

2.3.1 The WGC for $p$-form gauge fields

As mentioned, it seems only natural to extend the Weak Gravity Conjecture to general $p$-form gauge fields, as was argued in [5]. We assume that $p$-forms appear as

$$\frac{1}{2g} \int F_{p+1} \wedge \ast F_{p+1}. \quad (2.19)$$

The statement as given in [10] is:

<table>
<thead>
<tr>
<th>$p$-Form Weak Gravity Conjecture</th>
</tr>
</thead>
<tbody>
<tr>
<td>In $d$ dimensions, for each Abelian $p$-form gauge field with coupling $g$, there must be a $(p-1)$-dimensional object (a $(p-1)$-brane) with tension $T_p$ that carries integer charge $Q$ under this gauge field and satisfies</td>
</tr>
<tr>
<td>$$\frac{p(d-p-2)}{d-2} T_p^2 \leq g^2 Q^2 M_d^{d-2}, \quad (2.20)$$</td>
</tr>
<tr>
<td>where $M_d$ is the $d$-dimensional Planck mass.</td>
</tr>
</tbody>
</table>

We will not discuss the precise nature of the prefactor and are content with seeing that (2.20) reduces to the precise form of the electric WGC given in (2.4) if we set $p = 1$ and $d = 4$, which corresponds to the case of a point-particle with mass $m = T_1$ in four-dimensional spacetime that couples to a $U(1)$-gauge field. In that case, the inequality (2.20) reads

$$\frac{1}{2} m^2 \leq g^2 Q^2 M_4^2. \quad (2.21)$$

With $M_p = M_4$, we recover (2.4).

The conjecture is supported by the same argument as the one for the Weak Gravity Conjecture for a single $U(1)$: We demand that extremal black branes should be able to decay. There are some values of $p$ where this fails though, namely $p = 0$ and $p \geq d - 2$, and we need to address them separately:

i) A $d-1$ form is non-dynamical and a $d$-form couples to a $(d-1)$-brane which is space-time filling.
2.3 Generalized Conjectures

ii) While the $p = 0$ case is particularly interesting since it corresponds to axions and could yield a falsifiable prediction for the QCD axion. The obvious candidates for $(-1)$-dimensional objects are instantons that couple to these axions with the inverse decay constant playing the role of the gauge coupling. Evidence for this relation comes from string theory and we will come back to this issue in section 5.

2.3.2 The WGC for axions

It was suggested in [5] that the WGC can be extended to axions and instantons that couple to them in the following way:

<table>
<thead>
<tr>
<th>Weak Gravity Conjecture for Axions</th>
</tr>
</thead>
<tbody>
<tr>
<td>For any axion with decay constant $f$ there must be an instanton with action $S_E$ coupling to the axion such that $S_E \lesssim \frac{1}{f} M_p$ is satisfied.</td>
</tr>
</tbody>
</table>

We see that the euclidean instanton action $S_E$ is analogue to the mass “$m$” while the inverse of the axion decay constant $f$ plays the role of the coupling “$g$”. That this statement is a consequence of the standard WGC can be derived from T-dualities in string theory, where particles are mapped to instantons and vice versa [8].

One needs $S_E > 1$ in order to have $e^{-S_E} < 1$ for an instanton and hence, we conclude that the axion decay constant is at most of order of the Planck mass,

$$f \lesssim M_p.$$  \hspace{1cm} (2.23)

This is very interesting inasmuch as it provides a potentially falsifiable prediction for the QCD axion.

2.3.3 The WGC for multiple $U(1)$s

In [9], the Weak Gravity Conjecture was extended from $U(1)$ to a product gauge group of several $U(1)$s. Before we give the statement, note that the WGC can be rephrased in a slightly different way: Consider a $U(1)$ gauge theory with coupling
\section*{Weak Gravity Conjecture}

$g$ and particles of mass $m_i$ and charge $q_i$. We define the (dimensionless) ratios
\begin{equation}
 z_i := Q_i \frac{M_p}{m_i}, \quad \text{where} \quad Q_i := \sqrt{2} g q_i. \tag{2.24}
\end{equation}

Then, the electric WGC (2.4) is the statement that there exists a species $k$ such that
\begin{equation}
 1 \leq z_k. \tag{2.25}
\end{equation}

Now, let us extend the gauge group to a product $\prod_{a=1}^{n} U(1)_a$ with couplings $g^a$ such that each particle $m_i$ carries charges $q_i^a$. Similar to before, we define the ratios
\begin{equation}
 z_i^a := Q_i^a \frac{M_p}{m_i}, \quad \text{where} \quad Q_i^a := \sqrt{2} q_i^a g^a. \tag{2.26}
\end{equation}

For $i$ fixed, $q_i^a$ and $z_i^a$ are vectors of $SO(n)$ and we write $q_i$ and $z_i$ respectively.

It turns out that the “most obvious” generalization of the WGC to the present case, namely to conjecture that there is a species $k$ such that $|\vec{z}_k| > 1$, is not sufficient. However, demanding that all vectors $\vec{z}_i$ have $|\vec{z}_i| > 1$ is too strict. To determine the proper condition, we consider a black hole with charge $\vec{Q}$ and mass $M$ and - following the same line of reasoning as in 2.2 - demand that it be able to decay into a state with $n_i$ particles of species $i$ for $i = 1, ..., n$. Due to charge conservation,
\begin{equation}
 \vec{Q} = \sum_i \vec{Q}_i \tag{2.27}
\end{equation}

\footnote{That is, the index $i$ labels the species, each consisting of particles and antiparticles of mass $m_i$ and charge $q_i$ and $-q_i$ respectively.}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{example.png}
\caption{Example with $n = 2$ that satisfies the convex hull condition.}
\end{figure}
and with $\vec{Z} = \vec{Q} M_p / M$ and the vectors $\vec{z}_i$ as defined above, it follows that

$$\vec{Z} = \sum_i \vec{Q}_i M_p / M = \frac{1}{M} \sum_i n_i m_i \vec{z}_i. \quad (2.28)$$

Thus, we can interpret $\vec{Z}$ as weighted average of the charge vectors $\vec{z}_i$. From energy conservation it follows that the mass $M$ has to account for at least the masses of the decay particles, i.e.

$$1 > \frac{1}{M} \sum_i n_i m_i. \quad (2.29)$$

Hence, the black hole charge vector $\vec{Z}$ lies in the convex hull spanned by the vectors $\pm \vec{z}_i$. Since for an extremal black hole $|\vec{Z}| = 1$, this translates to the requirement that the unit ball must be enclosed by the complex hull. This motivates the following conjecture:

**Weak Gravity Conjecture for Several $U(1)$s**

Let $\{m_i\}$ be a number of particles carrying charge $\vec{q}_i = (q^a_i)$ under a product gauge group $\prod_{a=1}^N U(1)_a$ with couplings $g^a$ and vectors $\vec{Q}_i$ be defined via $Q^a_i = \sqrt{2} q^a_i g^a$. Then, the convex hull spanned by the (dimensionless) charge-to-mass ratios $\vec{z}_i = \vec{Q}_i M_p / m_i$ encloses the unit ball.

Fig. 8 illustrates a situation with two $U(1)$s where the conjecture is satisfied. Note that although $|\vec{z}_3| < 1$, the unit ball is enclosed in the convex hull. Likewise, one can easily think of a situation where all charge vectors $\vec{z}_i$ have $|\vec{z}_i| > 1$ that would still violate the conjecture: Take for example $n = 2$ and assume there are two species with $\vec{z}_{1,2}$ orthogonal to each other. Then, we can take their lengths slightly bigger than one but such that their convex hull intersects the unit disk. Since the boundary of the unit disk consists of extremal black hole solutions, this would correspond to a situation where stable black hole remnants exist that render the theory unphysical.

### 2.3.4 Gauge-Scalar Weak Gravity Conjecture

Finally, we come to an extension of the weak gravity conjecture that takes not only gauge and gravitational but also scalar forces into consideration. This is going to play a major role in the main part of the thesis where it is elaborated in more detail. Here - for the sake of simplicity - we will restrict to a situation where the
WGC particle couples to a single scalar field. In that case, it was proposed in [11] that the WGC bound (2.4) needs to be altered in the following way:

Gauge-Scalar Weak Gravity Conjecture

If the WGC particle is coupled to a scalar field with coupling $\mu$, the WGC bound is modified to

$$m^2 + \mu^2 M_p^2 \leq g^2 M_p^2$$

(2.30)

where absorbed the charge $q$ in the definition of $g$.

It is easy to give a physical interpretation to (2.30). We consider a fermion $\psi$ as WGC particle whose mass $m = m(\phi)$ is parameterized by a scalar $\phi$ as well as a single gauge-field $A_\mu$ under which the WGC particle is charged. Expanding the scalar field about its VEV, $\phi = \langle \phi \rangle + \delta \phi$ introduces a coupling to the WGC particle: We have $m(\phi) = m(\langle \phi \rangle) + \partial_\phi m(\langle \phi \rangle) \delta \phi$ and the Lagrangian contains a term

$$\mathcal{L} \supset \partial_\phi m \delta \phi \bar{\psi} \psi.$$  

(2.31)

Similarly, if the WGC particle is itself a (complex) scalar $\varphi$, the term $m^2(\phi)\varphi \varphi^*$ gives rise to a coupling

$$\mathcal{L} \supset 2m \partial_\phi m \delta \phi \varphi \varphi^*$$  

(2.32)

in the Lagrangian. We stick with the fermion case where

$$\mathcal{L} \supset m \bar{\psi} \psi + QA_\mu \gamma^\mu \bar{\psi} \psi + \partial_\phi m \delta \phi \bar{\psi} \psi.$$  

(2.33)
2.3 Generalized Conjectures

Clearly, this gives rise to the following tree-level interactions between two WGC particles via gravity, the gauge field and the scalar. Associated with these are forces of the form

$$F \sim g^2 r^{-2},$$

(2.34)

where $g$ is the respective coupling. These are attractive for gravity and the scalar and repulsive for the gauge field. Thus, (2.30) can be read as the statement that the gauge repulsion exceeds the combined forces of gravitational and scalar attraction. This was already sketched in figure 3.

We will establish in section 4.3 that equality holds for BPS states. Due to the different signs, in this case the sum over all forces vanishes: Two similar BPS particles put next to each other do not feel any force. For gravity to truly be the weakest force, we need that in addition to (2.30), the scalar interaction must exceed the gravitational force:

**Scalar Weak Gravity Conjecture**

If the WGC particle is coupled to a scalar field with coupling $\mu$, then

$$|\mu| M_p > m.$$  

(2.35)

Note that with $\mu = \partial_\phi m$, this is a differential equation that is easily integrated:

$$m \sim e^{-\frac{\phi}{M_p}}.$$  

(2.36)

Interestingly, this connects the WGC with another quantum-consistency condition proposed in [4] which we discuss as last conjecture about quantum gravity before turning to Calabi-Yau compactification.
2.3.5 Swampland Conjecture

It is well-known that the moduli space of a consistent quantum theory of gravity (this is true in string theory but assumed to be a general feature of quantum gravity) is parameterized by the expectation values of massless scalar fields. Hence, we can talk about the geometry of the moduli space by defining a metric via the kinetic terms of these scalars. A single point (as depicted in fig. 9) corresponds to a certain low-energy effective action and Ooguri and Vafa conjectured that displacements from such a point in the moduli space of a quantum theory of gravity lead to an infinite tower of states that become light exponentially fast. Distances in the moduli space can be defined as shortest geodesics with respect to the metric mentioned above.
2.3 Generalized Conjectures

Swampland Conjecture

Let $\mathcal{M}$ denote the moduli space of a quantum theory of gravity and let the distance $d(p, q)$ between two points of $\mathcal{M}$ be defined as shortest geodesic between them. Then, for any $p_0 \in \mathcal{M}$, the interval

$$\{d(p, p_0) | p \in \mathcal{M}\}$$

(2.37)

is not bounded from above and the theory at $p$ has an infinite tower of states with mass of order

$$m \sim e^{-\alpha d(p, p_0)}$$

(2.38)

with some $\alpha > 0$.

The conjecture implies that as the distance diverges, the low-energy effective theory breaks down due to the appearance of an infinite tower of light states. Hence, the theory corresponding to a particular point $p_0$ only makes sense in some finite region around this point.
3 Calabi-Yau compactification

In the thesis, we will mostly be dealing with the low-energy actions of Type IIA and IIB string theory. These are formulated in ten spacetime dimensions. Since this is in contradiction with our every-day experience, six of these dimensions must be such that they are not detectable in experiment. A possible way to resolve this issue is requiring the extra-dimensions to be small and compact such that they are invisible above a certain length scale. This is illustrated in fig. 10: Identifying two opposite sides of a rectangle yields a cylinder. Doing the same with the two remaining sides, one arrives at a torus which looks like a single point if we take its radii small enough. Before turning to the Type II theories, we will briefly review Kaluza-Klein reduction and discuss the moduli spaces of Calabi-Yau manifolds.

\[ \mathbb{R}^2 \text{ is curled up by identifying } (x, y) \equiv (x, y + 2\pi R) \]

\[ \mathbb{R} \times S^1 \text{ is closed to a torus by identifying } (x, y) \equiv (x + 2\pi r, y + 2\pi R) \]

\[ \text{shrinking} \quad \text{shrinking} \quad \text{point-like} \]

**Figure 10:** Compactified two-dimensional surface looks like a single point from low-dimensional perspective.

3.1 Kaluza-Klein reduction

In the Kaluza-Klein ansatz, one assumes that spacetime $\mathcal{M}$ has product structure\(^4\)

\[ \mathcal{M} = \mathcal{M}^{1,3} \times Y, \tag{3.1} \]

where $\mathcal{M}^{1,3}$ is some maximally symmetric space with four non-compact dimensions that represents our observed world - e.g. Minkowski space - and $Y$ is a compact

\(^4\)More generally, one can consider a so-called *warped product* but we will assume $\mathcal{M}$ to be a direct product.
manifold called internal or compactification manifold. We denote the coordinates on $\mathcal{M}^{1,3}$ by $x$ and those on the internal manifold by $y$. The ansatz (3.1) corresponds to having a metric of the form

$$G_{MN}(x,y) = \begin{pmatrix} g^{(4)}_{\mu\nu}(x) & 0 \\ 0 & g^{(6)}_{mn}(y) \end{pmatrix}. \quad (3.2)$$

The field content of IIA/IIB supergravity consists - apart from the metric - of $p$-forms. The equation of motion of a $p$-form $\hat{C}_p$ is

$$d \ast d \hat{C}_p = 0 \quad (3.3)$$

and together with the gauge condition

$$d \ast \hat{C}_p = 0, \quad (3.4)$$

this can be written as

$$\Delta^{(10)} \hat{C}_p = 0, \quad (3.5)$$

where the Laplacian is defined as $\Delta^{(10)} = (d + d^\dagger)^2$. In order to obtain a four-dimensional effective theory, one expands such a $p$-form $\hat{C}_p$ into a sum

$$\hat{C}_p(x,y) = C^k(x)\varphi_k(y), \quad (3.6)$$

where the $C^k$ and $\varphi_k$ are fields on $\mathcal{M}^{(1,3)}$ and $Y$ respectively. With the compactification ansatz (3.1), the ten-dimensional Laplacian decomposes as

$$\Delta^{(10)} = \Delta^{(4)} + \Delta^{(6)}. \quad (3.7)$$

In order for the four dimensional field to remain massless, we need

$$\Delta^{(4)} \hat{C}_p = 0 \quad (3.8)$$

and consequently

$$\Delta^{(6)} \varphi_k = 0. \quad (3.9)$$

This tells us that the expansion is one in terms of harmonic forms on the internal manifold.
3.2 Calabi-Yau requirement

So far, it is not clear what to chose as a compactification manifold, since we posed no further restrictions. But surely, the obtained four-dimensional theory would depend on this choice in a crucial way. The easiest ansatz $Y = \mathbb{T}^6$, i.e. taking the six-torus as compactification manifold, does not yield an appealing theory from a phenomenological point of view, as it leaves all ($\mathcal{N} = 4$ or $\mathcal{N} = 8$) supersymmetry unbroken in four dimensions. Still - despite the fact that so far, no traces of supersymmetry have been observed in experiment - it seems reasonable [21] to expect that at least some supersymmetry survives compactification to a four-dimensional theory. In particular, one can look for a model that possesses $\mathcal{N} = 1$ supersymmetry in four dimensions at high energies which are low compared to the compactification scale. A striking piece of evidence for this assumption is the resulting unification of the three gauge-couplings at about $10^{16}$ GeV in such a supersymmetric extension of the standard model as illustrated in fig. 11. What exactly does the requirement of unbroken supersymmetry imply for the structure of the compactification manifold $Y$? Spontaneous symmetry breaking is associated with non-vanishing vacuum expectation values: Consider a SUSY charge $Q$ with SUSY parameter $\varepsilon$. Having unbroken supersymmetry means (see e.g. [22] on this) that

$$\bar{\varepsilon} Q |0\rangle = 0. \quad (3.10)$$

\[ \text{Figure 11: Running couplings in minimal supersymmetric extension of standard model.} \]
In order to have SUSY preserved by the vacuum, we therefore need all supersymmetry variations to vanish in the vacuum,

$$\langle \delta \epsilon \Phi \rangle = \langle 0 | [\bar{\epsilon} Q, \Phi] | 0 \rangle = 0 \quad \text{for all fields } \Phi.$$  \hspace{1cm} (3.11)

Note that the variation $\delta b$ of a boson is fermionic and thus does not have a non-zero vacuum expectation value. Hence, we need only demand $\langle \delta f \rangle = 0$ for fermions.

In Type II string theory, the SUSY charges form two independent superalgebras. Associated with each of these is a ten-dimensional Majorana-Weyl spinor $\hat{\epsilon}^i$ with $i = 1, 2$. The gravitino which is present in both Type IIA and IIB transforms as

$$\delta \hat{\psi}_A^i = \nabla_A \hat{\epsilon}^i + \cdots$$  \hspace{1cm} (3.12)

where $\langle \cdots \rangle = 0$ in the absence of fluxes which we do not consider. Thus, the condition for unbroken SUSY is

$$\langle \nabla_A \hat{\epsilon}^i \rangle = 0,$$  \hspace{1cm} (3.13)

i.e. $\hat{\epsilon}^i$ must be \textit{covariantly constant} with respect to the background metric. Clearly, this restricts the allowed compactification manifolds$^5$. We will examine what this implies. In Type IIA the two spinors transform in the $16$ and $16'$ of $SO(1, 9)$, while in IIB both transform in the chiral representation $16$. With the ansatz (3.1), $SO(1, 9)$ decomposes as

$$SO(1, 9) \to SO(1, 3) \times SO(6)$$  \hspace{1cm} (3.14)

and we see that on $Y$, the spinors have two pieces that transform$^6$ as $4$ and $\bar{4}$ of $SU(4) \cong SO(6)$. If the spinors are covariantly constant on $Y$, they are left invariant upon parallel transport around any closed loop. But this is just another way to say that they transform as a singlet under the holonomy group$^7$

$$\text{Hol}(Y) \subset SO(6) \cong SU(4).$$  \hspace{1cm} (3.15)

There are different possibilities for $\text{Hol}(Y)$. Let us consider $\text{Hol}(Y) = SU(3)$ where $4$ decomposes into $3 + 1$ and - in a consequence - we arrive at one covariantly

---

$^5$Consider e.g. the two-sphere $S^2$. It is well-known that one cannot define a vector field that vanishes nowhere on $S^2$. But a covariantly constant vector field that vanishes at one point vanishes everywhere.

$^6$See e.g. [24].

$^7$We assume $Y$ to be orientable. Otherwise, $\text{Hol}(Y) \subset O(6)$. 

---
constant spinor of each chirality. To make this clear, we write (for Type IIA)
\[\varepsilon^1 = \varepsilon^1_+ \otimes \eta_+ + \varepsilon^1_- \otimes \eta_-,
\varepsilon^2 = \varepsilon^2_+ \otimes \eta_- + \varepsilon^2_- \otimes \eta_-,\]
(3.16)
The spinors \(\varepsilon^1_+\) and \(\varepsilon^2_-\) are two independent Weyl-spinors in four dimensions and for each covariantly constant \(\eta_+\), we get \(4 + 4\) supercharges in four dimensions - one for each component of \(\varepsilon^1_+\). Since we found one \(\eta_+\), this corresponds to \(N = 2\) SUSY in four dimensions.

We saw that the existence of exactly one covariantly constant spinor means reducing the holonomy of our compactification manifold to \(SU(3)\). For Kähler manifolds, this is the requirement to be Calabi-Yau, which we discuss in detail in appendix A. See in particular definition (A.13) from where we quote that a compact Kähler \(n\)-fold is Calabi-Yau iff it has holonomy \(\text{Hol} \subset SU(n)\). We did not yet establish that the internal manifold is Kähler, though. As discussed in the appendix, we need to show that the manifold is complex and has a closed Kähler form. For the former, it suffices to construct an almost complex structure with vanishing Nijenhuis tensor field. By means of the covariantly constant spinor, we build a bilinear
\[\mathcal{J}^m \equiv i\varepsilon^1_+ \gamma_{mp} g^{pn} \eta_+ = -i\varepsilon^1_- \gamma_{mp} g^{pn} \eta_-,
\]
(3.17)
where \(\gamma_{mn} = \frac{1}{2}[\gamma_m, \gamma_n]\) and \(\gamma^m\) are the internal gamma matrices. With help of the Fierz transformation formula (see e.g. [1]), one finds
\[\mathcal{J}^m \mathcal{J}^n_p = -\delta^m_n,\]
(3.18)
that is, the compactification manifold is almost complex with
\[\mathcal{J} = \mathcal{J}^m_n dx^m \otimes \frac{\partial}{\partial x^n}\]
(3.19)
the almost complex structure. Since \(\eta_+\), \(\eta_-\) and the metric are covariantly constant, \(\mathcal{J}\) is also covariantly constant and consequently, the Nijenhuis tensor vanishes:
\[N^m_{np} = \mathcal{J}^l_{[n} \partial_{l|p]}^m - \mathcal{J}^l_{[n} \partial_{l|}^m = 0.\]
This implies that \(Y\) is complex\(^8\). Finally, we can - according to (A.15) - define complex coordinates \(z^i, \bar{z}^\bar{i}\) such that the metric is Hermitian. Bearing in mind that

\(^8\)See (A.13) and below.
the metric is also covariantly constant, one sees that the Kähler form

$$J := i g_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

(3.21)

is closed. We briefly recapitulate: In order to maintain a minimal amount of supersymmetry, the six-dimensional compactification space is required to be a Kähler manifold with closed Kähler form and $SU(3)$ holonomy - in short: A Calabi-Yau threefold.

### 3.3 Reduction on the Calabi-Yau

We found that in Kaluza-Klein reduction, ten-dimensional fields are expanded in harmonic forms on the internal manifold. As discussed in the appendix, these are in one-to-one correspondence with elements of the cohomology groups (see (A.34)) and therefore counted by the Hodge numbers which - for a Calabi-Yau - take the form shown in fig. 14.

To make clear how this works in practice, consider as an example a three-form $\hat{C}_3$ with components $\hat{C}_{LMN}$ in $d = 10$. Then, $\hat{C}_{\mu\nu\sigma}$ does not carry any Calabi-Yau indices and thus is a scalar from the perspective of the compactification manifold. Likewise, $\hat{C}_{\mu\nu i}$ is a $(1,0)$-form (which does not exist on the internal space), $\hat{C}_{ij\bar{k}}$ a $(2,1)$-form and so on. This leads us to the correspondence

$$\hat{C}_{\mu\nu\sigma} \leftrightarrow H^{0,0}(Y), \quad \hat{C}_{ijk} \leftrightarrow H^{3,0}(Y), \quad \hat{C}_{\mu ij} \leftrightarrow H^{1,1}(Y),$$

$$\hat{C}_{ij\bar{k}} \leftrightarrow H^{2,1}(Y), \quad \hat{C}_{i\bar{j}\bar{k}} \leftrightarrow H^{1,2}(Y)$$

(3.22)

and

$$\hat{C}_{\mu\nu i} \leftrightarrow H^{1,0}(Y) = \emptyset, \quad \hat{C}_{\mu ij} \leftrightarrow H^{2,0}(Y) = \emptyset$$

(3.23)

since $h^{1,0} = 0 = h^{1,1}$. We will follow this scheme when performing the compactification of Type IIA and IIB.

In the appendix, a complex basis for $H^{2,1}(Y)$ is defined consisting of the $(2,1)$-forms $\{\eta_a\}$ with $a = 1, \ldots, h^{2,1}$. Another choice [25] is to consider a basis for the whole space

$$H^3(Y) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}.$$  

(3.24)

Note that $h^{3,0} = 1 = h^{0,3}$ (the corresponding cohomology groups only consist of $\Omega$ and $\bar{\Omega}$ respectively) and hence, we can chose a real basis $\{\alpha_{\hat{a}}, \beta_{\hat{a}}\}$ with $\hat{a} =$
3.4 Moduli spaces of Calabi-Yau threefolds

A Calabi-Yau three-fold $Y$ with given Hodge numbers is not uniquely determined. Instead, we can consider perturbations

$$g \rightarrow g + \delta g \quad (3.29)$$

The various basis forms of the different cohomology groups are listed in table 1. The relations (3.25) are preserved under the symplectic group $\text{Sp}(2h^{2,1} + 2)$, i.e. under transformations

$$\begin{pmatrix} \beta \\ \alpha \end{pmatrix} \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \quad (3.26)$$

with

$$A^T D - C^T B = 1 = (A^T D - C^T B)^T \quad (3.27)$$

and

$$A^T C = (A^T C)^T, \quad B^T D = (B^T D)^T. \quad (3.28)$$

We will make use of this structure in order to define a symplectic section that serves as projective coordinates on the Calabi-Yau moduli space.

### Table 1: Bases for cohomology groups of $Y$.

<table>
<thead>
<tr>
<th>Cohomology</th>
<th>Basis</th>
<th>Defined</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H^{1,1}(Y)$</td>
<td>$\omega_A$</td>
<td>in (A.35)</td>
</tr>
<tr>
<td>$H^{2,2}(Y)$</td>
<td>$\tilde{\omega}^A$</td>
<td>in (A.35)</td>
</tr>
<tr>
<td>$H^{2,1}(Y)$</td>
<td>$\eta_a$</td>
<td>below (A.35)</td>
</tr>
<tr>
<td>$H^{1,2}(Y)$</td>
<td>$\tilde{\eta}_a$</td>
<td>below (A.35)</td>
</tr>
<tr>
<td>$H^3(Y)$</td>
<td>$(\alpha^a, \beta^b)$</td>
<td>in (3.25)</td>
</tr>
</tbody>
</table>

0, 1, ..., $h^{2,1}$ satisfying

$$\int_Y \alpha^a \wedge \alpha^b = 0 = \int_Y \beta^a \wedge \beta^b, \quad \frac{1}{V_0} \int_Y \alpha^a \wedge \beta^b = \delta^b_a. \quad (3.25)$$
of its Kähler metric that leave it Ricci-flat. In the following discussion of this, we follow [26] and [27]. In order to still have $R_{mn}(g + \delta g) = 0$ and thus maintain the Calabi-Yau property, the variations $\delta g$ need to satisfy the Lichnerowicz equation

$$\nabla^k \nabla_k \delta g_{mn} + 2 R^p_{\ m \ n \ q} \delta g_{pq} = 0.$$ (3.30)

In complex coordinates, this splits into two independent equations for perturbations $\delta g_{ij}$ with mixed indices and such with pure indices, $\delta g_{ij}$ and $\delta g_{i\bar{j}}$, respectively.

1. The mixed variations correspond to a real $(1,1)$-form $i \delta g_{ij} dy^i \wedge d\bar{y}^j \in H^{1,1}(Y)$ (3.31) which allows us to expand

$$(g_{ij} + \delta g_{ij})(x, y) = -iv^A(x)(\omega_A)_{ij}(y), \quad A = 1, ..., h^{1,1},$$ (3.32)

where $\{\omega_A\}$ is the basis of $H^{1,1}(Y) \cong \text{Harm}^{1,1}(Y)$. Since the metric is directly related to the Kähler form, (A.24), we have $J = v^A \omega_A$ and call the real scalars $v^A$ Kähler moduli. Note that since $J \wedge J$ is a volume form$^9$, that is

$$\int_{\mathcal{M}_k} J > 0$$ (3.33)

for $k = 1, 2, 3$ and any complex $k$-dimensional submanifold of $Y$, these determine the volume of the internal manifold. In general, the metric deformations which preserve (3.33) form a cone as illustrated in the figure on the right: If the equation holds for $J$, then it holds for any $rJ$ with $r > 0$. Together with the $h^{1,1}$ real scalars $b^A$ arising from the expansion of the two-form $\hat{B}_2$ which appears together with the metric in Type II string theory, $\hat{B}_2(x, y) = B_2(x) + b^A(x)\omega_A(y)$, we define $h^{1,1}$ complex scalar fields

$$t^A := b^A + iv^A$$ (3.34)

forming the so-called complexified Kähler cone $\mathcal{M}^{ks}$.

$^9$See (A.25)
2. Since there are no \((2,0)\)-forms on a Calabi-Yau, the pure variations cannot be expanded directly. Instead, they correspond to a complex \((2,1)\)-form

\[
\Omega_{ijk}g^{km}\delta g_{ml}dy^i \wedge dy^j \wedge dy^k \in H^{2,1}(Y)
\] (3.35)

where \(\Omega\) is the holomorphic \((3,0)\)-form. We expand

\[
\Omega_{ijk}g^{km}\delta g_{ml} = \bar{z}^a(\bar{\eta}_a)_{ijk}, \quad a = 1, \ldots, h^{1,2}
\] (3.36)

or

\[
\delta g_{ij} = \frac{i}{\|\Omega\|^2} \bar{z}^a(\bar{\eta}_a)_{ik\ell} \Omega^\ell_j, \quad \|\Omega\|^2 := \frac{1}{3!} \Omega_{ijk}g^{ij}g^{km}\bar{\Omega}_{in\bar{n}}
\] (3.37)

in terms of \(h^{2,1}\) complex scalar fields \(\bar{z}^a\) and the basis \(\bar{\eta}_a\) for \(H^{1,2}(Y)\). There are two things that need to be noted at this point. First, we have

\[
dz^1 \wedge dz^2 \wedge dz^3 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3 = i^3 3!dx^1 \wedge dx^2 \wedge dx^3 \wedge dy^1 \wedge dy^2 \wedge dy^3
\] (3.38)

in coordinates \(z^i = x^i + iy^{i\,10})\). Second, since the components \(\Omega_{ijk}\) are antisymmetric, we need to have

\[
\Omega_{ijk}(z) = f(z)\varepsilon_{ijk}
\] (3.39)

for some holomorphic (and nowhere vanishing) function \(f\) and \(\varepsilon_{ijk}\) the epsilon tensor. Hence,

\[
\|\Omega\|^2 = |f|^2(\sqrt{g})^{-1}
\] (3.40)

and with this

\[
\int \Omega \wedge \bar{\Omega} = \int \frac{1}{3!} |f|^2 \varepsilon_{ijk} \varepsilon_{lmn} dz^i \wedge dz^j \wedge dz^k \wedge d\bar{z}^l \wedge d\bar{z}^m \wedge d\bar{z}^n
\]

\[
= \frac{i^3 3!}{3!} \int \|\Omega\|^2 d^6x \sqrt{q}
\]

\[
= -i\mathcal{V}\|\Omega\|^2
\] (3.41)

or

\[
\|\Omega\|^2 = \frac{i}{\mathcal{V}} \int_Y \Omega \wedge \bar{\Omega}.
\] (3.42)

\[\text{\footnotesize{\textsuperscript{10}}We write } d^6x \text{ for } dx^1dx^2dx^3dy^1dy^2dy^3.\]
Deformations (3.37) usually yield a metric which does no longer satisfy (A.19) and thus fails to be Hermitian. By a suitable coordinate transformation, the metric can be put in a form where the mixed-index components again vanish. But such a transformation is not holomorphic and this metric is thus Hermitian which respect to new complex coordinates, i.e. another complex structure [26].

The corresponding scalars $z^a$ are therefore called complex structure moduli and we denote the space they span by $\mathcal{M}^{cs}$.

Together, these scalars span the geometric moduli space of the Calabi-Yau, which locally is a product

$$\mathcal{M} = \mathcal{M}^{cs} \times \mathcal{M}^{ks}$$

of complex structure and Kähler structure moduli space respectively.

The most general metric [26] one can write for $\mathcal{M}$ is

$$ds^2 = -\frac{1}{2V} \int d^6x \sqrt{g} g^{ik} g^{jl} (\delta_{ij} \delta_{k\bar{l}} + \delta_{il} \delta_{jk} - \delta_{B_i} \delta_{B_j \bar{k}})$$

with $V$ the volume of the Calabi-Yau.

### 3.5 Complex structure moduli space

The part of the metric (3.44) corresponding to the complex structure moduli is

$$2G_{ab} z^a \bar{z}^b := -\frac{1}{2V} \int_Y d^6x \sqrt{g} g^{ij} g^{\bar{k}\bar{l}} \delta_{ij} \delta_{k\bar{l}}$$

and using the expansion (3.37) as well as (3.38) again, we have

$$2G_{ab} z^a \bar{z}^b = -\frac{2i}{V} \|\Omega\|^2 \int_Y \eta_a \wedge \bar{\eta}_b z^a \bar{z}^b,$$

that is,

$$G_{ab} = \frac{-i}{V} \|\Omega\|^2 \int_Y \eta_a \wedge \bar{\eta}_b.$$

We will work in the real basis defined in (3.25) and denote the three-cycles dual to $\{\alpha_a, \beta^a\}$ by $\{A^a, B_a\}$, i.e. one has

$$A^a \cap B_b = \delta^a_b = -B_b \cap A^a$$
and

\[ \int_{A^\alpha} \alpha_b^\dagger = \sqrt{V_0} i^b = - \int_{B^b} \beta^\dagger. \] (3.49)

In terms of this basis, \( \Omega \) can be expanded as

\[ \Omega = Z^a \alpha_a - F^b \beta^b \] (3.50)

with the periods

\[ Z^a = \frac{1}{\sqrt{V_0}} \int_{A^a} \Omega, \quad F^a = \frac{1}{\sqrt{V_0}} \int_{B^a} \Omega. \] (3.51)

The coordinates \( Z^a \) are actually projective because \( \Omega \) is homogeneous of degree one,

\[ (Z^0, Z^1, ..., ) \cong (\lambda Z^0, \lambda Z^1, ...), \] (3.52)

which allows us to chose

\[ z^a = \frac{Z^a}{Z^0}. \] (3.53)

This is discussed in great detail in B.3.2. The expansion

\[ \partial_{z^a} \Omega = k_a \Omega + i \eta_a \] (3.54)

which is derived in C.1.1 can be used to define a Kähler potential \( K^{cs} \) for the metric (3.47) via

\[ e^{-K^{cs}} := \frac{i}{V_0} \int_{Y} \Omega \wedge \bar{\Omega} = \frac{V}{V_0} \| \Omega \|^2 \] (3.55)
since
\[ \partial_a \bar{\partial}_a K^{cs} = - \partial_a \left( \frac{1}{\int_Y \Omega \wedge \bar{\Omega}} \int_Y \Omega \wedge (\bar{k}_b \bar{\Omega} - i \bar{\eta}_b) \right) \]
\[ = \frac{1}{(-i V \| \Omega \|^2)^2} \int_Y (k_a \Omega + i \eta_a) \wedge \bar{\Omega} \int_Y \Omega \wedge (\bar{k}_b \bar{\Omega} - i \bar{\eta}_b) \]
\[ - \frac{1}{-i V \| \Omega \|^2} \int_Y (k_a \Omega + i \eta_a) \wedge (\bar{k}_b \bar{\Omega} - i \bar{\eta}_b) \]
\[ = k_a \bar{k}_b - k_a \bar{k}_b - \frac{i}{V \| \Omega \|^2} \int_Y \eta_a \wedge \bar{\eta}_b \]
\[ = G_{ab}. \] (3.56)

Plugging in the expansion (3.50) for \( \Omega \),
\[ \frac{i}{V_0} \int_Y \Omega \wedge \bar{\Omega} = \frac{i}{V_0} \int_Y (Z^a \alpha_a - \mathcal{F}_b \beta^b) \wedge (\bar{Z}^c \alpha_c - \mathcal{F}_d \beta^d) \]
\[ = i (Z^a \mathcal{F}_a - \bar{Z}^a \mathcal{F}_a), \] (3.57)
we find that this expression is equal to the symplectic product introduced in B.3.2,
\[ e^{-K^{cs}} = -i \langle v, \bar{v} \rangle \text{ where } v = \left( Z^a \mathcal{F}_a \right). \] (3.58)

Hence, the Kähler metric can be written as the Kähler and symplectic covariant expression (B.37):
\[ G_{ab} = i \langle \nabla_a V, \nabla_b \bar{V} \rangle \] (3.59)

Here, \( V = e^{K^{cs}} v \) and the Kähler covariant derivatives are introduced in (B.34).
The \( k_a \) appearing in the expansion (3.54) can be determined explicitly from
\[ \partial_a \int_Y \Omega \wedge \bar{\Omega} = -i V_0 e^{-K^{cs}} \partial_a K^{cs} \] (3.60)
and
\[ \partial_a \int_Y \Omega \wedge \bar{\Omega} = \int_Y (k_a \Omega + i \eta_a) \wedge \bar{\Omega} = i V_0 e^{-K^{cs}} k_a. \] (3.61)
3.6 Kähler moduli space

From this we conclude

\[ k_a = -\partial_a K^{cs}, \]
\[ \partial_a \Omega = -(\partial_a K^{cs}) + i\eta_a \]  

(3.62)

and can therefore use the Kähler covariant derivative to write

\[ i\eta_a = \nabla_a \Omega \]  

(3.63)

and thus the metric on the complex structure moduli space as

\[ G_{ab} = -\frac{i}{V||\Omega||^2} \int \nabla_a \Omega \wedge \nabla_b \bar{\Omega} = -\frac{\int \nabla_a \Omega \wedge \nabla_b \bar{\Omega}}{\int \Omega \wedge \bar{\Omega}} \]  

(3.64)

### 3.6 Kähler moduli space

The metric on the Kähler moduli space is

\[ G_{AB} = \frac{1}{4V} \int_Y \omega_A \wedge * \omega_B \]  

(3.65)

which we derive in C.1.2. Using the volume form (A.25), we define

\[ K := \int_Y J \wedge J \wedge J = 6V \]  

(3.66)

and

\[ K_{ABC} := \int_Y \omega_A \wedge \omega_B \wedge \omega_C, \ K_{AB} := K_{ABC} v_C, \ K_A := K_{AB} v_B. \]  

(3.67)

Note that with this notation, \( K = K_A v^A \). Using the fact [28] that

\[ *\omega_A = -J \wedge \omega_A + \frac{3K_A}{2K} J \wedge J, \]  

(3.68)

the metric \( G_{AB} \) on \( \mathcal{M}^{ks} \) can then be written as

\[ G_{AB} = \frac{3}{2} \left( \frac{K_{AB}}{K} - \frac{3K_A K_B}{2K^2} \right). \]  

(3.69)
It has a Kähler potential\(^{11}\)

\[
K^{ks} := -\ln \frac{4}{3} \mathcal{K}. \quad (3.70)
\]

To see this, note that we have \(\partial_{\bar{t}A} = -\partial_{tA} = -\frac{1}{2i} \partial_{vA}\) on a function \(f(t^A) = f(v^A)\). Thus,

\[
\partial_{\bar{t}B} \mathcal{K} = \frac{i}{2} \partial_{vB} \mathcal{K}_{CDE} v^C v^D v^E = \frac{3i}{2} \mathcal{K}_B, \quad \partial_{tA} \mathcal{K}_B = \frac{1}{2i} \partial_{vA} \mathcal{K}_{BCD} v^C v^D = \frac{1}{i} \mathcal{K}_{AB}. \quad (3.71)
\]

Using this, we have

\[
\partial_{tA} \partial_{\bar{t}B} \left(-\ln \frac{4}{3} \mathcal{K}\right) = -\partial_{tA} \left( \frac{3i}{2\mathcal{K}} \mathcal{K}_B \right) = -\frac{3i}{2} \left( \frac{-1}{\mathcal{K}^2} \frac{3}{2i} \mathcal{K}_A \mathcal{K}_B + \frac{1}{\mathcal{K}} \frac{1}{i} \mathcal{K}_{AB} \right) = -\frac{3}{2} \left( \frac{\mathcal{K}_{AB}}{\mathcal{K}} - \frac{3}{2\mathcal{K}} \mathcal{K}_A \mathcal{K}_B \right), \quad (3.72)
\]

which shows that \(K^{ks}\) indeed is a Kähler potential for the metric. We define the inverse \(\mathcal{K}^{AB}\) via

\[
\mathcal{K}_{AB} \mathcal{K}^{BC} = \delta^C_A. \quad (3.73)
\]

which implies

\[
\mathcal{K}_B \mathcal{K}^{BC} = \mathcal{K}_{BDE} \mathcal{K}^{BC} = v^D \mathcal{K}^C = v^C. \quad (3.74)
\]

The inverse metric \(G^{AB}\) can be written as

\[
G^{AB} = -\frac{2\mathcal{K}}{3} \left( \mathcal{K}^{AB} - 3 \frac{v^A v^B}{\mathcal{K}} \right) \quad (3.75)
\]

\(^{11}\)We included the prefactor of \(\frac{4}{3}\) for later convenience.
3.7 Mirror symmetry

as can be seen by direct computation:

$$G_{AB}G^{BC} = \left( \mathcal{K}_{AB} - \frac{3}{2} \mathcal{K}_A \mathcal{K}_B \right) \left( \mathcal{K}^{BC} - 3 \frac{v^B v^C}{\mathcal{K}} \right)$$

$$= \delta^C_A + \frac{\mathcal{K}_A v^C}{\mathcal{K}} \left( -\frac{3}{2} - 3 + \frac{9}{2} \right)$$

$$= \delta^C_A. \quad (3.76)$$

The Kähler manifold actually is of a certain kind type which is discussed in detail in appendix B.3.2. Such $K^{ks}$ can be expressed in terms of a prepotential $\mathcal{F}$,

$$e^{-K^{ks}} = i(\bar{X}^\hat{A} \mathcal{F}_\hat{A} - X^\hat{A} \bar{\mathcal{F}}_\hat{A}), \quad (3.77)$$

which is defined as

$$\mathcal{F} := -\frac{1}{3!} \mathcal{K}_{ABC} \frac{X^A X^B X^C}{X^0}, \quad \mathcal{F}_\hat{A} := \partial_{X^\hat{A}} \mathcal{F}, \quad (3.78)$$

with coordinates $X^\hat{A} := (1, t^A)$. For explanations on the projective nature of the coordinates and a more general discussion of special Kähler manifolds we again refer to the appendix on special geometry. In C.1.3, we show that (3.77) indeed is a prepotential, i.e.

$$i(\bar{X}^\hat{A} \mathcal{F}_\hat{A} - X^\hat{A} \bar{\mathcal{F}}_\hat{A}) = \frac{4}{3} \mathcal{K} = 8 \mathcal{V} = e^{-K^{ks}}. \quad (3.79)$$

We sum up metric and Kähler potential for the two moduli spaces in table 2 for later reference. There is an interesting connection between the moduli spaces which we will exploit later. Hence, there is one topic to be discussed before turning to the main part.

3.7 Mirror symmetry

For a Calabi-Yau manifold $Y$ (we only consider threefolds), mirror symmetry states the following:

- there is a so-called mirror Calabi-Yau $\tilde{Y}$ with even and odd cohomologies identified. That is, their Hodge numbers are related via

$$h^{1,1}(\tilde{Y}) = h^{2,1}(Y), \quad h^{2,1}(\tilde{Y}) = h^{1,1}(Y) \quad (3.80)$$

which amounts to reflecting the Hodge numbers in the Hodge diamond (see fig. 14) along the diagonal.
### Table 2: Metrics and Kähler potentials for moduli spaces.

- The complex structure and Kähler moduli spaces of the mirror Calabi-Yau $\tilde{Y}$ are identified with the Kähler and complex structure moduli spaces of $Y$,

$$\mathcal{M}^{ks}(\tilde{Y}) = \mathcal{M}^{cs}(Y), \quad \mathcal{M}^{cs}(\tilde{Y}) = \mathcal{M}^{ks}(Y), \quad (3.81)$$

and the action resulting from compactification of Type IIA supergravity on some Calabi-Yau $Y$ is identical to the action of Type IIB supergravity compactified on the mirror manifold $\tilde{Y}$. In particular, one can also assume a cubic prepotential

$$\mathcal{F} := -\frac{1}{3!} \mathcal{K}^{ABC} \frac{X^A X^B X^C}{X^0}, \quad \mathcal{F}_A := \partial_{X^A} \mathcal{F}, \quad (3.82)$$

for the complex structure moduli space by switching to the mirror Calabi-Yau. We will not go into any more detail about mirror symmetry but refer to [29] for more information.

Having acquired the tools necessary for dealing with the moduli spaces of Calabi-Yaus, we can now begin the main part of the thesis.
4 A SCALAR WGC FOR TYPE IIB PARTICLES

4.1 Supersymmetric black holes

The argument we gave in support of the Weak Gravity Conjecture relied on the requirement that charged extremal black holes should be able to decay. Since compactification of Type II supergravity gives rise to $\mathcal{N} = 2$ SUGRA in $d = 4$ and in SUGRA, gravity is coupled to various scalars $\phi^a$, we will consider extremal black hole solutions of $\mathcal{N} = 2$ supergravity in presence of scalar fields [11]. The latter enter the action of a theory containing $U(1)$ gauge fields $V^A$ via symmetric real functions $\mathcal{I}_{AB}(\phi)$ and $\mathcal{R}_{AB}(\phi)$. The bosonic part of the action takes the generic form [30]

$$S = \int \ast \mathcal{R} - G_{ab} d\phi^a \wedge \ast d\phi^b + \frac{1}{2} \mathcal{I}_{AB} F^A \wedge \ast F^B + \frac{1}{2} \mathcal{R}_{AB} F^A \wedge F^B$$  \hspace{1cm} (4.1)$$

where $F^A = dV^A$ and the metric $g$ - entering implicitly via the Hodge-$\ast$ - is a function of the fields $\phi$.

4.1.1 Electromagnetic duality

The theory enjoys a symmetry that is analogous to the well-known\textsuperscript{12)\,} duality of Maxwell’s electromagnetism, i.e. the interchange of $F \leftrightarrow \ast F$. Obviously, this preserves the set of vacuum Maxwell equations

$$dF = 0, \quad dG = 0$$ \hspace{1cm} (4.2)$$

where the dual field strength $G$ is related to $F$ via

$$G = \frac{\delta}{\delta F} \int \frac{1}{2} F \wedge \ast F = \ast F.$$ \hspace{1cm} (4.3)$$

In the theory described in (4.1), the dual field strengths $G_A$ are defined similarly but mix the forms $F^A$ and their Hodge duals:

$$G_A := \frac{\delta S}{\delta F^A} = \mathcal{I}_{AB} * F^A + \mathcal{R}_{AB} F^A$$ \hspace{1cm} (4.4)$$

with Hodge-dual\textsuperscript{13)\,}

$$*G_A = \mathcal{I}_{AB} F^A - \mathcal{R}_{AB} * F^B.$$ \hspace{1cm} (4.5)$$

\textsuperscript{12)\,}See e.g. [31] for an extensive treatment.

\textsuperscript{13)\,}Note that $*^2 = -1$ when acting on even forms on the Calabi-Yau-threefold.
With this definition, we can write the Bianchi identities and equation of motion similar to (4.2) as

\[ dF^A = 0, \quad dG_A = 0 \]  

(4.6)

and look for a symmetry that preserves this set of equations as well as the definition \( G_A = \delta S / \delta F^A \). The latter requirement restricts [23] the allowed transformations to elements of the symplectic group,

\[ \begin{pmatrix} F^A \\ G_A \end{pmatrix} \rightarrow S \begin{pmatrix} F^A \\ G_A \end{pmatrix}, \]

(4.7)

with \( S \in \text{Sp}(2h, \mathbb{R}) \). Here, \( h \) denotes the number of vector fields \( V^A \), i.e. \( A = 1, ..., h \). Since we are considering charged, spherically symmetric and asymptotically flat black-hole solutions, we define the magnetic and electric charges \((p^A, q_A)\) by the integrals

\[ p^A := \frac{1}{4\pi} \int_{S^2} F^A, \quad q_A := \frac{1}{4\pi} \int_{S^2} G_A \]

(4.8)

performed over a sphere at infinity.\(^{14}\) By definition,

\[ \Gamma := \begin{pmatrix} p^A \\ q_A \end{pmatrix} \]

(4.9)

transforms as symplectic vector.

A discussion of the structure underlying \( \mathcal{N} = 2 \) supergravity is given in appendix B.3. From there, we quote that the scalars parameterizing \( \mathcal{I}_{AB} \) and \( \mathcal{R}_{AB} \) are the \( h \) complex scalars \( z^A \) from the gauge multiplets to which the one-forms \( V^A \) belong. As discussed in the appendix, the manifold spanned by these scalars is endowed with special Kähler geometry. Hence, there is a symplectic section \((Z^A, \tilde{F}_A)\) with the \( h + 1 \) fields \( Z^A(z^A) \) serving as projective coordinates on the manifold and \( \tilde{F}_A \) are functions of these coordinates. The Kähler potential is given by the symplectic invariant expression

\[ K = i \langle v, \bar{v} \rangle \]

(4.10)

with the symplectic vector \( v \) defined in (B.22). The metric can be written as the Kähler and symplectic invariant expression

\[ G_{AB} = i \langle \nabla_A V, \nabla_B \bar{V} \rangle \]

(4.11)

\(^{14}\)Note that the convention differs slightly from the one used in (2.4).
4.1 Supersymmetric black holes

in terms of the Kähler covariant derivative (B.34) and

\[ V = e^{\frac{i}{4}K}v = e^{\frac{i}{4}K}\left(\tilde{Z}^A(z^A)\right) \] (4.12)

as shown in (B.38).

4.1.2 Superalgebras with central charge and BPS bound

Remember that for massive representations of the \( \mathcal{N} = 1, d = 4 \) SUSY-algebra, the anti-commutator of the supercharges in the rest frame takes the form [22]

\[ \{Q_\alpha, Q_\beta^\dagger\} = 2m\delta_{\alpha\beta} \] (4.13)

where \( m \) is the mass of the states and \( \alpha, \beta = 1, ..., 4 \) label the Majorana spinor components. For extended supersymmetry, i.e. \( \mathcal{N} > 1 \), there are additional terms that are allowed by Lorentz invariance and which we thus include:

\[ \{Q_I^J, Q^I_J^\dagger\} = 2m\delta^{IJ}\delta_{\alpha\beta} + 2iZ^{IJ}\Gamma^0_{\alpha\beta} \] (4.14)

where \( I, J = 1, ..., \mathcal{N} \) with conserved quantities \( Z^{IJ} \). These are called central charges since they commute with all other generators in the superalgebra. Note that the matrix \( Z^{IJ} \) is necessarily antisymmetric which is why central charges can only appear for extended supersymmetries. We will only consider \( \mathcal{N} = 2 \) where the central charge matrix can be brought\(^{15}\) to the form

\[ Z^{IJ} = \begin{pmatrix} 0 & Z \\ -Z & 0 \end{pmatrix}. \] (4.15)

Since the left-hand side of (4.14) is non-negative, the eigenvalues of the right-hand side, namely \( 2m + 2Z \) and \( 2m - 2Z \), must also be non-negative. The resulting inequality

\[ m \geq |Z| \] (4.16)

is called BPS bound. The central charges are electric and magnetic charges coupling to the gauge fields [19]. States that saturate (4.16) are called BPS states.

\(^{15}\)To do so, perform unitary transformations \( Z \mapsto U^TZU \).
4.1.3 Relation for the central charge

We will now derive a relation for the central charge of the superalgebra that is going to play an important role in the next section.

First, note that the name graviphoton of the vector in the $N = 2$ supergravity multiplet (see table 5) stems from the fact that its field strength appears in the transformation of the gravitino [23]. It is given by the (scalar field dependent) combination of field strengths

$$e^{\frac{i}{2}K}(Z^A G_A - F_A F^A)$$

and therefore, we have the relation

$$Z = \int e^{\frac{i}{2}K}(Z^A G_A - F_A F^A) = e^{\frac{i}{2}K}Z^A q_A - F^A p_A$$

which identifies the central charge with the graviphoton charge at infinity. We write this in the concise form

$$Z = \langle V, \Gamma \rangle \text{ with } \Gamma = \left( \frac{p^A}{q_A} \right), \quad V = e^{\frac{i}{2}K} \left( \frac{X_A}{F_A} \right).$$

The sections $F_A$ and $X^B$ are related (see eq. (C.32)) via the gauge-coupling matrix $\mathcal{M}$ as

$$F_A = \mathcal{M}_{AB} X^B$$

and one finds

$$Z = X^A \left( q_A - \mathcal{M}_{AB} p^B \right), \quad \nabla_a Z = \nabla_a X^A (q_A - \mathcal{M}_{AB} p^B)$$

where we used that $\mathcal{M}$ is symmetric. Hence,

$$|Z|^2 + \nabla_a Z \nabla_b Z G^{ab} = (q_A - \mathcal{M}_{ACP} C)(q_B - \mathcal{M}_{BDP} D) \left( X^A \bar{X}^B + \nabla_a X^A \nabla_b \bar{X}^B G^{ab} \right)$$

and we identify the third factor on the right-hand side as $-\frac{1}{2} I^{-1}$. To simplify this equation further, we use the matrix introduced in [32]:

$$\mathcal{M} = \left( \begin{array}{cc} -(I + \mathcal{R} I^{-1} \mathcal{R})_{AB} & (\mathcal{R} I^{-1})^B_A \\ (I^{-1} \mathcal{R})^A_B & -(I^{-1})^A_B \end{array} \right)$$

- 40 -
and we define
\[ Q^2 := \frac{1}{2} \Gamma^T \mathcal{M} \Gamma. \]  \hfill (4.24)

In terms of these, (4.22) takes the form
\[ Q^2 = |Z|^2 + \nabla_a Z \bar{\nabla}_b \bar{Z} G^{a \bar{b}} \]  \hfill (4.25)

which we derive in C.2.1. Note that (4.25) is a statement about BPS states which are extremal with respect to (4.16), \( m = |Z| \). For a complex function \( f \), one has \( \partial |f| = \frac{1}{2if} (\partial f) \bar{f} \) and therefore, \( \partial |f| \bar{f} = \frac{1}{4} \partial f \bar{f} \). Hence, we can write this as
\[ Q^2 = m^2 + 4G^{a \bar{b}} \nabla_a m \bar{\nabla}_b m. \]  \hfill (4.26)

4.2 Gauge fields from reduction of Type IIB supergravity

Compactification of Type IIB supergravity on a Calabi-Yau three-fold gives rise to a number of \( U(1) \)s in the four-dimensional theory. As we will see, D3-branes wrapping three-cycles can be viewed as particles carrying charge under these gauge fields and thus should satisfy the weak gravity conjecture. The compactification of Type II supergravity follows mainly the discussions in [25, 33, 34, 35]. For an overview of compactification in the presence of background fluxes see [36]. A summary of results on Calabi-Yau manifolds and their moduli spaces and a short discussion of Type II SUGRA is given in the appendices A and B.

4.2.1 Action for Type IIB SUGRA

In appendix B, we discuss the ten-dimensional supergravities Type IIA and IIB which are the low-energy limits of the Type II string theories. From there we cite the action (B.14) where we now put hats on the ten-dimensional quantities:
\[ S_{\text{IIB}}^{(10)} = \frac{1}{2\kappa^2} \int e^{-2\phi} \left( * \tilde{R} + 4d\hat{\phi} \wedge *d\hat{\phi} - \frac{1}{2} \hat{H}_3 \wedge * \hat{H}_3 \right) - \frac{1}{4\kappa^2} \int \left( \hat{F}_1 \wedge * \hat{F}_1 + \hat{F}_3 \wedge * \hat{F}_3 + \hat{F}_5 \wedge * \hat{F}_5 \right) - \frac{1}{4\kappa^2} \int \hat{C}_4 \wedge \hat{H}_3 \wedge \hat{F}_3 \]  \hfill (4.27)
with the Kalb-Ramond 2-form $B_2$, the dilaton $\hat{\phi}$ and metric $\hat{g}$ from the NS-NS sector as well as the axion $\hat{C}_0$, 2-form $\hat{C}_2$ and 4-form $\hat{C}_4$ from the R-R sector and

$$
\hat{H}_3 = d\hat{B}_2, \\
\hat{F}_1 = d\hat{C}_0, \\
\hat{F}_3 = d\hat{C}_2 - \hat{C}_0 \hat{H}_3, \\
\hat{F}_5 = d\hat{C}_4 - \hat{C}_2 \wedge \hat{H}_3, 
$$

(4.28)

### 4.2.2 Expanding the fields

We make the Kaluza-Klein reduction ansatz discussed in section 3.1 and expand the fields (4.28) in terms of harmonic forms on the Calabi-Yau, i.e. in terms of the basis forms listed in table 1. We will discuss these fields in turn.

i) The axion $\hat{C}_0$ has no internal index and corresponds to $H^{0,0}(Y)$.

ii) The two-form $\hat{B}_2$ (and likewise $\hat{C}_2$) decomposes in $\hat{B}_{\mu\nu}$ corresponding to $H^{0,0}(Y)$ and $\hat{B}_{\mu i}$ corresponding to $H^{1,1}(Y)$. There are no further terms, see our discussion in section 3.3.

iii) The $\hat{C}_{\mu\nu}$ part of $\hat{C}_4$ is a four-form in $d = 4$ and therefore closed. Hence, it does not contribute to the action. The pieces $\hat{C}_{ijk}, \hat{C}_{ijk}, \hat{C}_{ij\bar{k}}$ and $\hat{C}_{ijk}$ correspond to $H^3(Y) = H^{3,0}(Y) \oplus H^{2,1}(Y) \oplus H^{1,2}(Y) \oplus H^{0,3}(Y)$ and the remaining $\hat{C}_{\mu\nu ij}$ and $\hat{C}_{ijk\bar{i}}$ to $H^{1,1}(Y)$ and $H^{2,2}(Y)$ respectively.

Accordingly, we expand

$$
B_2(x,y) = B_2(x) + b^A(x)\omega_A(y), \\
\hat{C}_2(x,y) = C_2(x) + c^A(x)\omega_A(y), \\
\hat{C}_4(x,y) = D^A_2(x) \wedge \omega_A(y) + \varrho_A(x)\hat{\omega}^A(y) + V^{\hat{a}}(x) \wedge \alpha_{\hat{a}}(y) - U_{\hat{a}}(x) \wedge \beta^A(y) 
$$

(4.29)

with

$$
\hat{a} = 0, \ldots, h^{2,1}, \\
\hat{A} = 0, \ldots, h^{1,1}, \\
 a = 1, \ldots, h^{2,1}, \\
 A = 1, \ldots, h^{1,1}. 
$$

(4.30)

From now on, we will not write the coordinates explicitly. The vectors $V^{\hat{a}}$ and $U_{\hat{a}}$ (as well as $D^A_2$ and $\varrho_A$) are actually not independent but related by the self-duality $F_5 = \ast F_5$ of the field strength $F_5$. We will keep the one-forms $V^{\hat{a}}$ and scalars $\varrho_A$. 

-- 42 --
4.2 Gauge fields from reduction of Type IIB supergravity

<table>
<thead>
<tr>
<th>Multiplet</th>
<th>(Massless) Field Content</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gravity multiplet</td>
<td>( (g_{\mu\nu}, V^0) )</td>
<td>1</td>
</tr>
<tr>
<td>Gauge multiplets</td>
<td>( (V^a, z^a) )</td>
<td>( h^{2,1} )</td>
</tr>
<tr>
<td>Hypermultiplets</td>
<td>( (b^A, c^A, v^A, \varrho_A) )</td>
<td>( h^{1,1} )</td>
</tr>
<tr>
<td></td>
<td>( (h_B, h_C, \phi, C_0) )</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 3: Type IIB multiplets in \( d = 4 \).

Defining \( H_3 := dB_2, F^\alpha := dV^\alpha \) and \( G_\hat{\alpha} := dU_\hat{\alpha} \) we have

\[
\begin{align*}
\hat{H}_3 &= H_3 + db^A \wedge \omega_A, \\
\hat{C}_2 &= H_3 + dc^A \wedge \omega_A, \\
\hat{C}_4 &= dD_2^A \wedge \omega_A + F^\alpha \wedge \alpha_\hat{\alpha} - G_\hat{\alpha} \wedge \beta^\hat{\alpha} + d\varrho_A \wedge \tilde{\omega}^A.
\end{align*}
\]

(4.31)

We are only interested in the action for the gauge fields \( V^\alpha \) and \( G_\hat{\alpha} \) which comes from the \( F_5 \wedge *F_5 \)-term in the action (4.27). Plugging in the above-mentioned expansion for the expression of the field strength \( F_5 \) in (4.28), we obtain

\[
\begin{align*}
\hat{F}_5 &= dD_2^A \wedge \omega_A + F^\alpha \wedge \alpha_\hat{\alpha} - G_\hat{\alpha} \wedge \beta^\hat{\alpha} + d\varrho_A \wedge \tilde{\omega}^A - (C_2 + c^A \omega_A) \wedge (H_3 + db^A \omega_A) \\
&= F^\alpha \wedge \alpha_\hat{\alpha} - G_\hat{\alpha} \wedge \beta^\hat{\alpha} + d\varrho_A \wedge \tilde{\omega}^A + (dD_2^A - C_2 \wedge db^A - c^A \wedge H_3) \wedge \omega_A \\
&\quad - c^A db^B \wedge \omega_A \wedge \omega_B.
\end{align*}
\]

(4.32)

4.2.3 Field content of Type IIB in four dimensions

As discussed in section 3, the four-dimensional theory possesses \( \mathcal{N} = 2 \) SUSY. Hence, the massless fields form three kinds of irreducible representations, namely a gravity multiplet, gauge multiplets and hypermultiplets [23]. Again, we will discuss these in turn.

i) The gravity multiplet contains the graviton \( g_{\mu\nu} \) and a one-form \( V^0 \).

ii) A gauge multiplet in \( \mathcal{N} = 2, d = 4 \) contains a one-form and a complex scalar. The only (remaining) one-forms are \( V^a \) which combine with the complex structure moduli \( z^a \).

iii) Hypermultiplets contain four real scalars. Hence, \( b^A \) and the Kähler moduli \( v^A \) combine with \( c^A \) and \( \varrho_A \) to \( h^{1,1} \) multiplets.
iv) The two-forms $B_2$ and $C_2$ are actually Poincaré dual to scalars $h_B, h_c$ [25] which form another hypermultiplet with the dilaton $\phi$ and $C_0$.

The various multiplets are collected in table 3.

4.2.4 Integrating on the Calabi-Yau

The non-vanishing terms in the integral over $\hat{F}_5 \wedge \ast \hat{F}_5$ containing the field strengths $F^a$ and $G_a$ are

$$I_1 = -\frac{1}{4\kappa^2} \int_Y (F^a \wedge \alpha_a) \wedge \ast (F^b \wedge \alpha_b - G_b \wedge \beta^b)$$

(4.33)

and

$$I_2 = -\frac{1}{4\kappa^2} \int_Y (-G_a \wedge \beta^a) \wedge \ast (F^b \wedge \alpha_b - G_b \wedge \beta^b).$$

(4.34)

We use the integrals from A.4 to get

$$4\kappa^2 I_1 = -F^a \wedge \ast F^b [\text{Im } \mathcal{M} + (\text{Re } \mathcal{M})(\text{Im } \mathcal{M})^{-1}(\text{Re } \mathcal{M})]_{\hat{a}\hat{b}}$$

$$+ F^a \wedge \ast G_b [(\text{Re } \mathcal{M})(\text{Im } \mathcal{M})^{-1}]_{\hat{a}}^b$$

$$4\kappa^2 I_2 = G_a \wedge \ast F^b ((\text{Re } \mathcal{M})(\text{Im } \mathcal{M})^{-1})_{\hat{a}}^b - G_a \wedge \ast G_b [(\text{Im } \mathcal{M})^{-1}]_{\hat{a}\hat{b}}$$

(4.35)

where we defined the four-dimensional coupling $\kappa^2 = \kappa^2/\mathcal{V}_0$. This can be written in a more compact way. For example,

$$I := \text{Im } \mathcal{M}^{-1}(\mathcal{M}F) \wedge \ast (\tilde{\mathcal{M}}F)$$

(4.36)

is the $F \wedge F$-term form above,

$$I = [(\text{Im } \mathcal{M})^{-1}]_{\hat{a}\hat{b}} (\text{Re } \mathcal{M} + i \text{ Im } \mathcal{M})_{ac} F^c (\text{Re } \mathcal{M} - i \text{ Im } \mathcal{M})_{bd} F^d$$

$$+ [(\text{Im } \mathcal{M})^{-1}]_{\hat{a}\hat{b}} (\text{Re } \mathcal{M})_{ac} (\text{Im } \mathcal{M})_{bd} F^c \wedge \ast F^d$$

$$= (\text{Re } \mathcal{M})_{ac} [(\text{Im } \mathcal{M})^{-1}]_{\hat{a}\hat{b}} (\text{Re } \mathcal{M})_{bd} F^c \wedge \ast F^d + (\text{Im } \mathcal{M})_{\hat{a}\hat{b}} F^c \wedge \ast F^d. $$

(4.37)

One can see that together with the remaining terms, this takes the form

$$\text{Im } \mathcal{M}^{-1}(G - \mathcal{M}F) \wedge \ast (G - \tilde{\mathcal{M}}F).$$

(4.38)

There are, of course, other non-vanishing terms arising from the $F_5 \wedge \ast F_5$ integral but we are not interested in these.
4.2 Gauge fields from reduction of Type IIB supergravity

4.2.5 Action for the gauge fields

We have not yet imposed the self-duality condition $F_5 = *F_5$. First, note that $F_5 = *F_5$ implies

\[
*F_\hat{a} \wedge *\alpha_\hat{a} - *G_\hat{a} \wedge *\beta_\hat{a} = * F_\hat{a} \wedge \left( (\text{Re } M)(\text{Im } M)^{-1}\hat{a} \hat{b} \right) +
\]

\[
[- \text{Im } M - (\text{Re } M)(\text{Im } M)^{-1}(\text{Re } M)]\hat{a}\hat{b}\hat{c} -
\]

\[
* G_\hat{a} \wedge \left( [((\text{Im } M)^{-1})\hat{a}\hat{b}\hat{c}] - [(\text{Re } M)(\text{Im } M)^{-1}]\hat{a}\hat{b}\hat{c} \right)
\]

\[
= F_\hat{a} \wedge \alpha_\hat{a} - G_\hat{a} \wedge \beta_\hat{a}
\] (4.39)

and by equating coefficients\(^{16}\)

\[
*G = \text{Re } M * F - \text{Im } M F
\]

\[
G = \text{Re } M F + \text{Im } M * F.
\] (4.40)

The self-duality can then be imposed via the equation of motion for $G_\hat{a}$ by adding the term

\[
\frac{1}{2} F_\hat{a} \wedge G_\hat{a}
\] (4.41)

as a Lagrange multiplier to the Lagrangian for the fields strengths $F_\hat{a}$ and $G_\hat{a}$:

\[
\mathcal{L}_{F_a} = \frac{1}{4} (\text{Im } M)^{-1}(G - M F) \wedge *(G - M F) + \frac{1}{2} F_\hat{a} \wedge G_\hat{a}.
\] (4.42)

Variation with respect to $G_\hat{a}$ shows that (4.40) is now implemented:

\[
\delta_G \mathcal{L}_{F_a} = \frac{1}{4} (\text{Im } M)^{-1}(*(G - M F) + *(G - M F)) + \frac{1}{2} F
\]

\[
= \frac{1}{2} (\text{Im } M)^{-1}(*G - \text{Re } M * F) + \frac{1}{2} F = 0,
\] (4.43)

i.e.

\[
*G = \text{Re } M * F - \text{Im } M F.
\] (4.44)

\(^{16}\text{See footnote 13 to eq. (4.5).}\)
Eliminating $G_a$ in $\mathcal{L}_{F^a}$ via its equation of motion yields

$$
\mathcal{L}_{F^a} = \frac{1}{4} (\text{Im} \mathcal{M})^{-1} (\text{Re} \mathcal{M} \mathcal{F} + \text{Im} \mathcal{M} \mathcal{F}^* - \mathcal{M} \mathcal{F} - \bar{\mathcal{M}} \mathcal{F}^*) \\
+ \frac{1}{2} \mathcal{F} \wedge (\text{Re} \mathcal{M} \mathcal{F} + \text{Im} \mathcal{M} \mathcal{F}) \\
= \frac{1}{4} (\text{Im} \mathcal{M})^{-1} (\text{Im} \mathcal{M} \mathcal{F} - \text{Im} \mathcal{M} \mathcal{F}^* \mathcal{F} - \bar{\mathcal{M}} \mathcal{F}^*) \\
+ \frac{1}{2} \mathcal{F} \wedge (\text{Re} \mathcal{M} \mathcal{F} + \text{Im} \mathcal{M} \mathcal{F}) \\
= \frac{1}{2} \text{Im} \mathcal{M}_{ab} \mathcal{F}^a \wedge \ast \mathcal{F}^b + \frac{1}{2} \text{Re} \mathcal{M}_{ab} \mathcal{F}^a \wedge \mathcal{F}^b
$$

(4.45)

which is a kinetic and a topological term for the gauge fields $V^a$. We note:

$$
S_{\text{gauge}} = \int \frac{1}{2} \text{Im} \mathcal{M}_{ab} \mathcal{F}^a \wedge \ast \mathcal{F}^b + \frac{1}{2} \text{Re} \mathcal{M}_{ab} \mathcal{F}^a \wedge \mathcal{F}^b.
$$

(4.46)

This is precisely what we had expected for the action of the gauge fields in $\mathcal{N} = 2$ supergravity - see section 4.1. By comparison with the action (4.1), we see that the roles of $\mathcal{I}$ and $\mathcal{R}$ are played by the imaginary and real part of the matrix $\mathcal{M}$.

4.3 Particles from D3-branes

As already mentioned, D3-branes wrapping three-cycles in Type IIB look like particles in the four-dimensional theory and it will turn out that these satisfy a modified version of the WGC.

We begin with the action [22, 37]

$$
S_{D3}^{(10)} = -\mu_3 \int_{D3} d^4 \xi e^{-\phi} \sqrt{-\det(G)} + \mu_3 \int_{D3} \mathcal{C}_4
$$

(4.47)

of a D3-brane coupled to the four-form $\mathcal{C}_4$ and wrapping a three-cycle $\mathcal{C}$. The brane tension $\mu_3$ is related\(^\text{17}\) to the ten-dimensional gravitational coupling as $\mu_3 = \sqrt{\pi/\kappa}$ and $G$ denotes the pullback of the spacetime metric onto the brane’s world-volume. Introducing the integer charges $q_a$ and $p^a$ by expanding $\mathcal{C}$ in terms of the three-cycles $\mathcal{A}^a$ and $\mathcal{B}_a$ dual to the three-forms $\alpha_a$ and $\beta^a$ (see 3.49),

$$
\mathcal{C} = q_a \mathcal{A}^a + p^a \mathcal{B}_a,
$$

(4.48)

\(^\text{17}\)For a $Dp$-brane we have $\mu_p = (\sqrt{\pi/\kappa})(2\pi\sqrt{\alpha'})^{3-p}$, see [22] eq. (13.3.7).
and using the expansion (4.29), the second integral in (4.47) reads

$$\mu_3 \int_{D3} \hat{C}_4 = \mu_3 \sqrt{V_0} \left( q_a \int V^a + p^\beta \int U_\beta \right).$$  \hspace{1cm} (4.49)

To compute the first integral, we perform a Weyl rescaling (B.7) with $e^{-\frac{\hat{\phi}}{2}}$ to get rid of the dilaton factor and with $(\mathcal{V}_0/\mathcal{V})^{-1/4}$ such that

$$-\mu_3 \sqrt{\frac{V_0}{\mathcal{V}}} \int_{D3} d^4 \xi \sqrt{-\text{det}(G)}.$$  \hspace{1cm} (4.50)

The volume of the cycle has a lower bound

$$\text{Vol}(\mathcal{C}) \geq \frac{1}{\sqrt{||\Omega||}} \left| \int_{\mathcal{C}} \Omega \right| = \sqrt{\frac{V}{V_0}} e^{\frac{1}{2} K_{cs}} \left| \int_{\mathcal{C}} \Omega \right|$$  \hspace{1cm} (4.51)

where we used (3.42) and the definition of the Kähler potential (3.55). Note that we did only choose a homology class in (4.48) and thus cannot give more than this bound for the volume. We can specify the cycle as to minimize this volume\(^{18}\) in which case an equal sign holds. This is true for a BPS state. We will come back to this issue later and assume for now that the volume is minimized. Using the expansion (3.50) of the holomorphic three-form, we can perform the integration over $\mathcal{C}$,

$$\int_{\mathcal{C}} \Omega = \int_{\mathcal{C}} (Z^a \alpha_a - F^b \beta_b) = \sqrt{V_0} \int d\tau (p_a Z^a - q^\beta F_\beta),$$  \hspace{1cm} (4.52)

to arrive at

$$-\mu_3 \sqrt{\frac{V_0}{\mathcal{V}}} \text{Vol}(\mathcal{C}) = -\mu_3 \sqrt{V_0} e^{\frac{1}{2} K_{cs}} \left| q_a Z^a - p^\beta F_\beta \right| \int d\tau.$$  \hspace{1cm} (4.53)

We have

$$\mu_3 = \frac{\sqrt{\pi}}{\kappa} = \frac{\sqrt{\pi}}{\kappa \sqrt{V_0}}$$  \hspace{1cm} (4.54)

\(^{18}\)Such a cycle is called supersymmetric special Lagrangian.
and are working in units where $\sqrt{\pi}/\kappa = 1$. Thus, the four-dimensional action is

$$S_{D3}^{(4)} = -e^{\frac{i}{2}K^{cs}} \left| q_{\hat{a}} Z^{\hat{a}} - p^{\hat{b}} F_{\hat{b}} \right| \int d\tau + q_{\hat{a}} \int V^{\hat{a}} + p^{\hat{a}} \int U_{\hat{a}}. \quad (4.55)$$

Clearly, this is the action of a particle with mass

$$m = e^{\frac{i}{2}K^{cs}} \left| q_{\hat{a}} Z^{\hat{a}} - p^{\hat{b}} F_{\hat{b}} \right| \quad (4.56)$$

charged under the gauge fields $V^{\hat{a}}$ and $U_{\hat{a}}$. One recognizes $m$ as the central charge (4.19). Hence, a brane wrapping a supersymmetric cycle gives rise to a particle that is extremal with respect to the BPS bound (4.16),

$$m = |Z|. \quad (4.57)$$

4.4 A scalar WGC for the particle

4.4.1 Applying the central charge relation

We found out that the central charge is related to the symplectic charges and gauge couplings via (4.25) and that a D3-brane wrapping a supersymmetric Lagrangian three-cycle looks like a particle that is extremal with respect to the BPS bound upon compactification to four dimensional spacetime. Hence, we can apply the central charge relation derived in the last section,

$$Q^2 = m^2 + 4 G^{a\bar{b}} \nabla_a m \nabla_{\bar{b}} m. \quad (4.58)$$

If we consider only electric charges and set

$$\Gamma = \begin{pmatrix} 0 \\ q_{\hat{a}} \end{pmatrix}, \quad (4.59)$$

the quantity $Q$ defined in (4.24) becomes

$$Q^2 = -\frac{1}{2} q_{\hat{a}} (I^{-1})^{\hat{a}\hat{b}} q_{\hat{b}} \quad (4.60)$$

where $I = \text{Im} \mathcal{M}$. In that case, $Q$ is the analogue to the electric charge $Q$ appearing in the electric weak gravity conjecture (2.4) and we will make this clear by explicit calculation from the prepotential. Before doing so, we should point out the following: Recall that we assumed that the brane is wrapping the three-cycle
such that it minimizes its volume and that only in this case the mass is extremal
with respect to the BPS bound. Dropping this assumption would lead to \( m \geq |Z| \)
but it is not obvious how the second term with derivatives of \( m \) would be affected. We
keep the volume minimized.

The equality (4.58) is in accordance with a generalization of the Gauge-Scalar
WGC as we presented it in (2.30) to situations with several gauge and scalar fields.
In particular - since the second term on the right-hand side is positive definite - it
implies the Weak Gravity Conjecture

\[
m^2 \leq Q^2. \tag{4.61}
\]

### 4.4.2 Explicit calculation from prepotential

Now, to get better understanding of (4.58), we explicitly carry out the calculation
in the prepotential formalism. To do so, we need an explicit expression of the
gauge-coupling matrix \( \mathcal{M} \). Recall\(^{19}\) that mirror symmetry allows us to assume
a cubic prepotential for the moduli in the following way: Let \( \hat{Y} \) be the mirror
Calabi-Yau of \( Y \). Since by mirror symmetry \( \text{dim}_\mathbb{C} H^{2,1}(Y) = \text{dim}_\mathbb{C} H^{1,1}(\hat{Y}) \)
and \( \text{dim}_\mathbb{C} H^{1,1}(Y) = \text{dim}_\mathbb{C} H^{2,1}(\hat{Y}) \), we interpret \( K^{ks} := K^{cs} \) as Kähler potential for
the Kähler moduli on \( \hat{Y} \) and likewise, \( \hat{X}^\hat{A} := Z^\hat{a}, \hat{F}^\hat{A} := F_\hat{a} \) as belonging to its
prepotential. We will drop the tilde in the following.

We only consider charges under the vectors in the gauge multiplet and hence
set \( q^\hat{A} = (0, q_A) \), \( p^\hat{A} = 0 \), such that

\[
m^2 = e^{K^{ks}} |q_A t^A|^2. \tag{4.62}
\]

We can now make use of an explicit expression for the gauge-coupling matrix \( \mathcal{M} \). This can be derived directly from the prepotential but takes some time and
effort while at the same time being not very illuminating. Hence, we perform the
calculation in the appendix and from there cite eq. (C.59):

\[
(\text{Im } \mathcal{M}^{-1})^{A\dot{B}} = -\frac{6}{K} \left( b^A \left( \frac{1}{3} G^{AB} b^B + b^A b^B \right) \right). \tag{4.63}
\]

Therefore,

\[
Q^2 = -\frac{1}{2} q_A (\text{Im } \mathcal{M}^{-1})^{A\dot{B}} q_B
= e^{K^{ks}} \left( q_A G^{AB} q_B + 4(q_A b^A)^2 \right). \tag{4.64}
\]

\(^{19}\)See section 3.7.
We can choose $t^A = b^A + i v^A$ to be purely imaginary\(^\text{20)\)} such that

$$Q^2 = e^{K_{ks}} q_A G^{AB} q_B,$$

$$m^2 = e^{K_{ks}} |q_A i v^A|^2.$$  \(4.65\)

Next, we compute the Kähler covariant derivatives

$$\nabla_A m = \partial_{i A} m + \frac{1}{2} (\partial_{i A} K^{ks}) m$$  \(4.66\)

that appear on the right-hand side of (4.58):

$$\nabla_A m = \frac{1}{2i} \left( \partial_{v^A} + \frac{1}{2} \partial_{v^A} K^{ks} \right) m = \frac{1}{2i} e^{\frac{1}{2} K^{ks}} (\partial_{v^A} + \partial_{v^A} K^{ks}) (q_A v^A).$$  \(4.67\)

With

$$\partial_{v^A} K^{ks} = -\frac{1}{K} \partial_{v^A} \mathcal{K}_{BCD} v^B v^C v^D = -\frac{3 K_A}{\mathcal{K}},$$  \(4.68\)

we find

$$\nabla_A m = \frac{1}{2i} e^{\frac{1}{2} K^{ks}} \left( q_A - \frac{3 K_A}{2 \mathcal{K}} q_B v^B \right).$$  \(4.69\)

We need the following expressions:

$$G^{AB} \mathcal{K}_A = -\frac{2 \mathcal{K}}{3} \left( \mathcal{K}^{AB} - \frac{3 v^A v^B}{\mathcal{K}} \right) \mathcal{K}_A$$

$$= -\frac{2 \mathcal{K}}{3} (v^B - 3 v^B)$$

$$= e^{-K^{ks}} v^B$$  \(4.70\)

and

$$G^{AB} \mathcal{K}_A \mathcal{K}_B = \frac{4}{3} \mathcal{K} v^A \mathcal{K}_B = \frac{4}{3} \mathcal{K}^2.$$  \(4.71\)

\(^{20)\)}This is because the metric $G$ does not depend on the axions.
4.4 A scalar WGC for the particle

With these, the right-hand side of (4.58) takes the form

\[ m^2 + 4G^{AB}\nabla_A m\nabla_B m = \frac{1}{4}e^{K_{ks}}(q_{A\nu}A)^2 + \]

\[ G^{AB}e^{K_{ks}} \left( q_A - \frac{3}{2}K_A q_C v_C \right) \left( q_B - \frac{3}{2}K_B q_D v_D \right) \]

\[ = e^{K_{ks}} \left[ (q_{A\nu}A)^2 + q_A G^{AB} q_B - 2\frac{4}{3}(q_{A\nu}A)^2 + \frac{9}{4}q_{A\nu}A) \right] \]

\[ = e^{K_{ks}} q_A G^{AB} q_B. \tag{4.72} \]

This is precisely what we found in (4.65) and thus we have shown that

\[ q_A G^{AB} q_B = e^{-K_{ks}} \left( m^2 + 4G^{AB}\nabla_A m\nabla_B m \right), \tag{4.73} \]

confirming eq. (4.58). In the next section, we will perform a similar calculation in order to establish a weak-gravity bound for an axion-instanton-pair.
5 A SCALAR WGC FOR TYPE IIA INSTANTONS

5.1 Axions from reduction of Type IIA supergravity

We now turn to Type IIA supergravity where compactification gives rise to axions $\xi^a$ and $\tilde{\xi}_a$ in the four dimensional theory. Unlike before, there is no self-duality that relates these two sets of fields.

Our starting point is the ten-dimensional low-energy action (B.11) that is obtained from eleven-dimensional supergravity as discussed in appendix B.1:

$$S_{IIA}^{(10)} = -\frac{1}{4\kappa^2} \int e^{-2\phi} \hat{H}_3 \wedge *\hat{H}_3 - \frac{1}{4\kappa^2} \int \hat{F}_2 \wedge *\hat{F}_2 - \frac{1}{4\kappa^2} \int \hat{F}_4 \wedge *\hat{F}_4 + \frac{1}{4\kappa^2} \int \hat{H}_3 \wedge C_3 \wedge d\hat{C}_3 + \frac{1}{2\kappa^2} \int e^{-2\phi} \left( R + 4d\phi \wedge *d\phi \right)$$

(5.1)

We recall the definitions

$$\hat{H}_3 = dB_2, \quad \hat{F}_2 = d\hat{A}_1, \quad \hat{F}_4 = d\hat{C}_3 - \hat{A}_1 \wedge \hat{H}_3$$

(5.2)

and proceed similar to the discussion of Type IIB in the last section.

5.1.1 Expanding the fields

By now, it should be clear how the four-dimensional fields arise. The forms are expanded as

$$\hat{A}_1 = A^0$$
$$\hat{C}_3 = C_3 + A^A \wedge \omega_A + \xi^a \alpha_a - \tilde{\xi}_a \beta^a$$
$$\hat{B}_2 = B_2 + b^A \omega_A,$$

(5.3)

where

$$a = 1, ..., h^{2,1}, \quad \hat{a} = 0, 1, ..., h^{2,1},$$
$$A = 1, ..., h^{1,1}, \quad \hat{A} = 0, 1, ..., h^{1,1}.$$

(5.4)

The corresponding field strengths are

$$d\hat{A}_1 = dA^0,$$
$$d\hat{C}_3 = dC_3 + dA^A \wedge \omega_A + d\xi^a \wedge \alpha_a - d\tilde{\xi}_a \wedge \beta^a,$$
$$\hat{H}_3 = H_3 + db^A \wedge \omega_A$$

(5.5)
Table 4: Type IIA multiplets in $d = 4$.

<table>
<thead>
<tr>
<th>Multiplet</th>
<th>(Massless) Field Content</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gravity multiplet</td>
<td>$(g_{\mu\nu}, V^0)$</td>
<td>1</td>
</tr>
<tr>
<td>Gauge multiplets</td>
<td>$(V^A, t^A)$</td>
<td>$h^{1,1}$</td>
</tr>
<tr>
<td>Hypermultiplets</td>
<td>$(z^a, \xi^a, \bar{\xi}^a)$</td>
<td>$h^{2,1}$</td>
</tr>
<tr>
<td></td>
<td>$(h_B, \phi, \xi^0, \bar{\xi}^0)$</td>
<td>1</td>
</tr>
</tbody>
</table>

with $H_3 := dB_2$. Since we are eventually interested in the axions $\xi^\hat{a}, \bar{\xi}^\hat{a}$, we take a closer look at the expansion of $\hat{F}_4$. Plugging in the above expressions this becomes

$$\hat{F}_4 = dC_3 + dA^A \wedge \omega_A + d\xi^{\hat{a}} \wedge \alpha_{\hat{a}} - d\bar{\xi}^{\hat{a}} \wedge \beta^{\hat{a}} - A^0 \wedge H_3 - A^0 \wedge db^A \wedge \omega_A. \quad (5.6)$$

5.1.2 Field content of Type IIA in four dimensions

Note that the three-form $C_3$ is dual to a constant in $d = 4$ and thus does not carry any degree of freedom. The remaining fields form the following multiplets

i) The gravity multiplet consists of the graviton $g_{\mu\nu}$ and the one-form $V^0$.

ii) The one-forms $V^A$ combine with the complex $t^A = b^A + iv^A$ to $h^{1,1}$ gauge multiplets.

iii) Complex structure moduli $z^a$ form $h^{2,1}$ hypermultiplets with the scalars $\xi^a$ and $\xi_a$.

iv) The two-form $B_2$ is dual to a scalar $h_B$ which combines in an additional hypermultiplet with the dilaton $\phi$, $\xi^0$ and $\bar{\xi}_0$.

For later reference, these multiplets are collected in table 4.

Now, we will derive the four-dimensional actions for the axions $\xi$ and $\bar{\xi}$. 

– 54 –
5.1 Axions from reduction of Type IIA supergravity

5.1.3 Integrating on the Calabi-Yau

The only non-vanishing terms in the integral of $\hat{F}_4 \wedge *\hat{F}_4$ are

$$I_1 := \int_Y (dC_3 - A^0 \wedge H_3) \wedge *(dC_3 - A^0 \wedge H_3),$$

$$I_2 := \int_Y (dA^A - A^0 \wedge db^A) \wedge \omega_A \wedge *[(dA^B - A^0 \wedge db^B) \wedge \omega_B],$$

$$I_3 := \int_Y (d\xi^\hat{a} \wedge \alpha^\hat{a} - d\tilde{\xi}^\hat{a} \wedge \beta^\hat{a}) \wedge *(d\xi^\hat{b} \wedge \alpha^\hat{b} - d\tilde{\xi}^\hat{b} \wedge \beta^\hat{b})$$ \hspace{1cm} (5.7)

and of these, only $I_3$ contributes a kinetic term for the axions:

$$I_3 = -d\xi^\hat{a} \wedge *d\xi^\hat{b}[(\text{Im } \mathcal{M} + (\text{Re } \mathcal{M})(\text{Im } \mathcal{M})^{-1}(\text{Re } \mathcal{M}))_{\hat{a}\hat{b}}$$

$$+ 2d\tilde{\xi}^\hat{a} \wedge *d\xi^\hat{b}[(\text{Re } \mathcal{M})(\text{Im } \mathcal{M})^{-1}]_{\hat{b}} - d\tilde{\xi}^\hat{a} \wedge *d\tilde{\xi}^\hat{b}[(\text{Im } \mathcal{M})^{-1}]_{\hat{a}\hat{b}}$$

$$= (\text{Im } \mathcal{M})^{-1\hat{a}\hat{b}}[d\tilde{\xi}^{\hat{a}} + \mathcal{M}_{\hat{a}\hat{c}}d\xi^{\hat{c}}] \wedge *[d\tilde{\xi}^{\hat{b}} + \bar{\mathcal{M}}_{\hat{b}\hat{d}}d\xi^{\hat{d}}].$$ \hspace{1cm} (5.8)

The kinetic part in the action for the axions $\tilde{\xi}^\hat{a}$ is therefore described by the inverse $\mathcal{I}^{-1} = (\text{Im } \mathcal{M})^{-1}$,

$$S_{\tilde{\xi}} = \int (\mathcal{I}^{-1})^{\hat{a}\hat{b}} d\tilde{\xi}^{\hat{a}} \wedge *d\tilde{\xi}^{\hat{b}}.$$ \hspace{1cm} (5.9)

The one for the axions $\xi^\hat{a}$ is

$$S_{\xi} = \int (\text{Im } \mathcal{M}^{-1})^{\hat{a}\hat{b}} \mathcal{M}_{\hat{a}\hat{c}}\bar{\mathcal{M}}_{\hat{b}\hat{d}}d\xi^{\hat{c}} \wedge *d\xi^{\hat{d}}$$

$$= \int (\text{Im } \mathcal{M}^{-1})^{\hat{a}\hat{b}}(\text{Re } \mathcal{M} + i \text{ Im } \mathcal{M})_{\hat{a}\hat{c}}(\text{Re } \mathcal{M} - i \text{ Im } \mathcal{M})_{\hat{b}\hat{d}}d\xi^{\hat{c}} \wedge *d\xi^{\hat{d}}$$

$$= \int \left[(\text{Im } \mathcal{M}^{-1})^{\hat{a}\hat{b}}(\text{Re } \mathcal{M}_{\hat{a}\hat{c}}\text{ Re } \mathcal{M}_{\hat{b}\hat{d}}) + \text{ Im } \mathcal{M}_{\hat{c}\hat{d}}\right]d\xi^{\hat{c}} \wedge *d\xi^{\hat{d}}$$ \hspace{1cm} (5.10)

which we write as

$$S_{\xi} = \int (RI^{-1}R + I)^{\hat{a}\hat{b}} d\xi^{\hat{a}} \wedge *d\xi^{\hat{b}}.$$ \hspace{1cm} (5.11)

The next section will focus on the axions $\tilde{\xi}$ in order to make the calculation easier.
5.2 Instantons from E2-Branes

We saw in section 4.3 that a D3-brane wrapping a three-cycle in Type IIB looks like a zero-dimensional object - a point particle - in four dimensions that carries charge under the one-forms which arise from dimensional reduction of the four-form coupling to the D3-brane. Further, it was shown that this particle satisfies a refined form of the Weak Gravity Conjecture.

In the present section, we consider a setting with a Euclidean E2-brane that couples to the three-form $\hat{C}_3$. Wrapping such E2-branes around appropriate cycles gives rise to $(-1)$-dimensional objects upon compactification to four dimensions. This is symbolized by the sphere in fig. 13 which from afar looks like a single point in spacetime. We will now derive the four-dimensional action for such a brane. Our starting point is

$$S_{E2}^{(10)} = -\mu_2 \int_{E2} d^3\xi e^{-\phi} \sqrt{-\det(G)} + \sqrt{2} \mu_2 \int_{E2} \hat{C}_3, \quad (5.12)$$

the action of an E2-brane coupled to the three-form $\hat{C}_3$ which (like the D3-brane in the last section) wraps a supersymmetric three-cycle

$$C = q_a A^a - p^b F_b. \quad (5.13)$$

---

21) An $E_p$-brane is a $p$-brane whose world-volume time is euclideanized. See e.g. [1].
This time, the integral is Euclidean and therefore yields the brane volume. We find:

\[ S_{E^2}^{(4)} = -e^{1/2 K_{ab}} \left[ q_a Z^a - p^b F_b \right] + q_a \xi^a + p^a \tilde{\xi}_a. \]  

(5.14)

This can be interpreted as an object localized (see the illustration in fig. 13) in spacetime - an instanton - with action

\[ S = e^{1/2 K_{ab}} \left| q_a Z^a - p^b F_b \right| \]  

(5.15)

coupled to the axions \( \xi^a \) and \( \tilde{\xi}_a \).

5.3 A scalar WGC for instantons and axions

We will set \( q_a = 0 \), i.e. consider charges

\[ \Gamma = \left( \begin{array}{c} p^b \\ 0 \end{array} \right), \quad p^b = (0, p^a), \]  

(5.16)

implying that the instanton couples to \( \tilde{\xi} \) only. Similar to the last section, we compute the expression

\[ S^2 + \nabla_a S \nabla_b S G^{ab}. \]  

(5.17)

5.3.1 Calculation

The prepotential and coordinates are related via \( F_a = M_{ab} Z^b \) (see eq. (C.32)) and therefore,

\[ S = e^{1/2 K} |p^a M_{ab}|. \]  

(5.18)

Here, the axions \( b^a = 0 \) were set to zero again such that

\[ M_{ab} = -\frac{1}{2} e^{-K} G_{ab}. \]  

(5.19)

The calculation goes through as before. We have

\[ \nabla_a S = \partial_a S + \frac{1}{2} (\partial_a K) S, \quad \text{where} \quad \partial_a S = \frac{1}{2} (\partial_a K) S + e^{1/2 K} \partial_a |p^a F_a|, \]  

(5.20)
and thus
\[ \nabla_a S \nabla_b S = \frac{1}{4} e^K \left( -\frac{3K_a}{2K} p^c \mathcal{M}_{cd} v^d + \mathcal{M}_{ac} p^c \right) \left( -\frac{3K_b}{2K} p^e \hat{\mathcal{M}}_{cd} v^d + \hat{\mathcal{M}}_{bd} p^d \right). \] (5.21)

Hence,
\[ 4 e^K G^{ab} \nabla_a S \nabla_b S = G^{ab} \left( G_{ac} - \frac{3K_a}{2K} G_{cd} v^c \right) \left( G_{be} - \frac{3K_b}{2K} G_{ef} v^f \right) p^c p^e \]
\[ = G_{ab} p^b - \frac{3}{4} \left( G_{ab} p^b v^b \right)^2 + \frac{9}{4} \left( G_{ab} p^b v^b \right)^2, \]
(5.22)
that is
\[ G^{ab} \nabla_a S \nabla_b S = \frac{1}{4} e^{-K} \left( G_{ab} p^b - \left( G_{ab} p^b v^b \right)^2 \right). \] (5.23)

In the last steps, we used
\[ G^{ab} K_a = \frac{4}{3} K v^b, \quad G^{ab} K_a K_b = \frac{4}{3} K^2, \] (5.24)
several times. The second term on the right-hand side of (5.23) is equal to minus \( S^2 \):
\[ S^2 = e^K \left| p^a \left( -\frac{i}{2} e^{-K} G_{ab} i v^b \right) \right|^2 \]
\[ = e^{-K} \frac{1}{4} \left( p^a G_{ab} v^b \right)^2. \] (5.25)

Putting all terms together,
\[ S^2 + G^{ab} \nabla_a S \nabla_b S = \frac{1}{4} e^{-K} p^a G_{ab} p^b. \] (5.26)

Remember that we found a kinetic term\(^{22}\)
\[ \int \left[ (\text{Im} \mathcal{M})^{-1} \right] \delta \xi_a \wedge *d\xi_b \] (5.27)

\(^{22}\)Note the appearance of the inverse \((\text{Im} M)^{-1}\) in contrast to what we had in the last section.
for the axions $\tilde{\xi}$ that couple to the instanton with charges $p$. Hence, the corresponding charge $Q^2$ is
\[
Q^2 = -\frac{1}{2} p_a (\text{Im } M)_{ab} p^b \\
= \frac{1}{4} e^{- K} p^a G_{ab} p^b
\]
(5.28)
This is equal to the right-hand side of (5.26) such that we finally arrive at the identity
\[
Q^2 = S^2 + G^{ab} \nabla_a S \bar{\nabla}_b S.
\]
(5.29)
Note that we established (5.29) as a statement about the axions $\tilde{\xi}$ by turning off the charges $q^{23)}$. However, it is sensible to assume that it holds also for the axions $\xi$. For these we found (see eq. (5.11)) a kinetic term
\[
(\mathcal{R} T^{-1} - I)_{ab} \, d\xi^a \wedge *d\xi^b
\]
(5.30)
which makes the calculation leading to (5.29) pretty awful. We will not carry it out but instead propose that (5.29) also holds if the instanton couples to $\xi$ or both types of axions, corresponding to having $q$ or both, $q$ and $p$ charges.

5.3.2 Result

Equation (5.29) is an extension of the Weak Gravity Conjecture (2.22) for axions to a situation with scalar fields present. For simplicity, we will now consider the case of a single scalar field $\phi$ and a single axion with decay constant $f$ coupled to the instanton. With $Q = 1/f$ and still working in units where $M_p = 1$, we then have
\[
1 = f^2 (S^2 + \partial_\phi S \bar{\partial}_\phi S).
\]
(5.31)
In particular - since the second term is positive definite - we recover the axion WGC
\[
1 \geq f S,
\]
(5.32)
that is, $f < 1$ if we assume $S > 1$.

\[23) This corresponds to the lower right part of $\mathcal{M}$ in (4.23).\]
We have established (5.29) for branes wrapping supersymmetric cycles. This is similar to the situation discussed in the last section where we saw that a D3-brane wrapping a supersymmetric three-cycle gives rise to a BPS particle. We argued that for two such particles, gauge, gravitational and scalar interaction would cancel, while for non-BPS states, the former would exceed the combined gravitational and scalar forces. Hence, it seems natural to propose that without supersymmetry, equation (5.29) becomes a bound

\[ Q^2 \geq S^2 + G^{ab} \nabla_a S \nabla_b S. \]

Motivated by this evidence, we propose that there is a general extension of the axion-instanton Weak Gravity Conjecture to situations with scalar fields which is analogous to the gauge-scalar Weak Gravity Conjecture and (reinstalling the Planck mass) takes the form

\[ M_p^2/f^2 \geq S^2 + |\partial_\phi S|^2 M_p^2. \]

Since the second term on the left-hand side of this relation is positive-definite, this lowers the bound \( f < M_p \) for the axion decay constant which follows from demanding \( e^{-S} < 1 \) for an instanton and we have the new bound

\[ f < M_p/\sqrt{1 + |\partial_\phi S|^2 M_p^2} \leq M_p. \]
6 CONCLUSION

We have discussed the Weak Gravity Conjecture in the presence of scalar fields and presented evidence from IIB supergravity where three-branes wrapping supersymmetric Lagrangian three-cycles look like (BPS) particles after compactification to four-dimensional spacetime. Such a particle is charged under the gauge fields arising from the expansion of the Type IIB four-form and saturates the Gauge-Scalar Weak Gravity Conjecture. If the particle carries charge under a single gauge field and with only one scalar field present, the bound takes the form

\[ m^2 + |\partial_m^2| M_p^2 \leq g^2 M_p^2. \]

For two BPS particles, this translates to the statement that the combined gravitational, gauge and scalar forces between them vanishes.

Motivated by this, we studied compactification of Type IIA string theory, where a Euclidean E2-brane wrapping a supersymmetric Lagrangian three-cycle looks like an instanton in the four-dimensional theory. Expansion of the Type IIA three-form gave rise to certain axions and we found a relation for these which is similar to the Gauge-Scalar Weak Gravity Conjecture:

\[ Q^2 = S^2 + G^{ab} \nabla_a S \nabla_b S \]

where \( S \) is the instanton action and \( Q^2 \) describes the coupling of these axions to the instanton. More precisely,

\[ Q^2 = \frac{1}{2} \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} -\left( I + R I^{-1} R \right) & RI^{-1} \\ I^{-1} R & -I^{-1} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \]

where the matrices \( I \) and \( R \) are defined in terms of the prepotential for the Calabi-Yau moduli.

We therefore argued that the axion-instanton Weak Gravity Conjecture might be extended to situations where scalar fields are present. More precisely, we proposed a bound of the form

\[ S^2 + |\partial_\phi S|^2 M_p^2 \leq M_p^2 / f^2 \]

relating instanton action \( S \), axion decay constant \( f \) and scalar field \( \phi \). While we gave a physical interpretation of the Gauge-Scalar Weak Gravity Conjecture in terms of forces, it is not clear to us how the additional \( \partial_\phi S \partial_\phi S \) should be understood in the case of axions and instantons. Since this term is positive definite, we argued that scalar fields actually lower the bound \( f < M_p \) that follows from the axion-instanton Weak Gravity Conjecture when demanding \( e^{-S} < 1 \) for the instanton action.
A Mathematical Preliminaries

While the reader is expected to be familiar with basic differential geometry [31], we review some of the mathematical definitions and results on complex manifolds that are used in the thesis. More detailed discussions on this can be found in [38, 39, 40] while a brief treatment of Kähler and Calabi-Yau manifolds is given in [27].

A.1 Complex manifolds

Simply put, a complex manifold is a manifold $M$ that allows us to define holomorphic functions $f : M \to \mathbb{C}$. To do so in a patch independent way, one needs the transition functions to be holomorphic. We define

**Definition A.1 (Complex manifold)**

A complex manifold is a differentiable manifold with a holomorphic atlas.

Note that a complex manifold necessarily has even real dimension since the coordinate functions are maps onto $\mathbb{C}^m$. Instead of pondering on complex manifolds, we will begin with differential manifolds that are not necessarily complex but admit a so-called almost complex structure which resembles the one that complex manifolds carry: While they might not even locally look like $\mathbb{C}^m$, their tangent spaces are complex. Eventually, we will think of complex manifolds as differentiable manifolds with almost complex structure and vanishing Nijenhuis tensor field. The reason we are choosing this approach is that it is the one that comes naturally when we discussing string compactification.

**Definition A.2 (Almost complex structure)**

A $(1, 1)$-tensor field $\mathcal{J}$ on a differentiable manifold $M$ satisfying

$$\mathcal{J}^2 = -\text{id} \quad (A.1)$$

is called almost complex structure. In that case, $(M, \mathcal{J})$ is called almost complex manifold.

Note that since $TM \otimes T^*M = \text{End}(TM)$, the field $\mathcal{J}$ defines an endomorphism $\mathcal{J}_p$ of every fiber $T_pM$. Writing $\mathcal{J}_p$ in local coordinates,

$$\mathcal{J}_p = \mathcal{J}_{\mu,\nu}(p)dx^\mu \otimes \frac{\partial}{\partial x^\nu}, \quad (A.2)$$

Note that since $TM \otimes T^*M = \text{End}(TM)$, the field $\mathcal{J}$ defines an endomorphism $\mathcal{J}_p$ of every fiber $T_pM$. Writing $\mathcal{J}_p$ in local coordinates,
condition (A.1) becomes

\[ J_\mu \sigma (p) J_\sigma \nu (p) = -\delta_\mu^\nu. \] (A.3)

Taking the determinant of (A.3), we find that almost complex manifolds need to have even real dimension. Not every \(2m\)-dimensional differentiable manifolds admits an almost complex structure, though. For example, the only spheres that do are \(S^2\) and \(S^6\) [39].

The almost complex structure now allows us to define (anti-)holomorphic vector fields as follows: We extend \(J_p\) to the complexification \(T_p M^\mathbb{C}\). Since \(J_p^2 = -\text{id}_{T_p M}\), the almost complex structure \(J_p\) has eigenvalues \(\pm i\) and we can therefore decompose

\[ T_p M^\mathbb{C} = T_p M^+ \oplus T_p M^- \] (A.4)

where the two disjoint spaces \(T_p M^\pm\) are spanned by the eigenvectors with eigenvalue \(\pm i\). This can be extended to the whole tangent bundle and we write

\[ T M^\mathbb{C} = T M^+ \oplus T M^- \] (A.5)

The complexification \(\Gamma(TM)^\mathbb{C}\) consists of the vector fields \(Z = X + iY\) with \(X, Y \in \Gamma(TM)\) and we define the complex conjugate \(\bar{Z} := X - iY\). With a projection

\[ \mathcal{P}^\pm := \frac{1}{2} (\text{id} \mp iJ), \] (A.6)

we can then decompose any vector field \(W \in \Gamma(TM)^\mathbb{C}\) as

\[ W = Z_1 + \bar{Z}_2 \] (A.7)

with \(Z_1 := \mathcal{P}^+ W\) and \(\bar{Z}_2 := \mathcal{P}^- W\).

**Definition A.3 ((Anti-)holomorphic vector field)**

A vector field \(Z = \mathcal{P}^+ Z\) is called holomorphic and a vector field \(\bar{Z} = \mathcal{P}^- \bar{Z}\) anti-holomorphic.

---

\(^{24}\)By the *complexification* \(V^\mathbb{C}\) of a real vector space \(V\) we mean the complex space spanned by the linear combinations \(a + ib\) with \(a, b \in V\) and all vector space operations defined in the obvious way. It has complex dimension \(\dim_{\mathbb{C}} V^\mathbb{C} = \dim_{\mathbb{R}} V\).
A.1 Complex manifolds

The decomposition of the complexified tangent bundle \((A.5)\) also defines the complexified cotangent bundle

\[ T^*M^C := T^*M^+ \oplus T^*M^- \]  

(A.8)

This gives us a way to define the space \(\Omega^{r,s}(M)\) of \((r,s)\)-forms on \(M\): The exterior product \(\Lambda^k T^*M^C\) decomposes as

\[ \Lambda^k T^*M^C = \Lambda^{0,k}M \oplus \Lambda^{1,k-1}M \oplus \ldots \oplus \Lambda^{k,0}M \]  

(A.9)

where

\[ \Lambda^{r,s}M := \Lambda^r T^*M^+ \otimes \Lambda^s T^*M^- \]  

(A.10)

**Definition A.4** \((r,s)\)-forms

A section of \(\Lambda^{r,s}M\) is called a \((r,s)\)-form on \(M\). We denote the set of \((r,s)\)-forms with \(\Omega^{r,s}(M)\).

For a complex manifold, the exterior derivative of a \((r,s)\) form can be decomposed \([40]\) as

\[ d\omega^{r,s} = \alpha^{r+1,s} + \beta^{r,s+1} \]  

(A.11)

which amounts to a decomposition \(d = \partial + \bar{\partial}\). We define:

**Definition A.5** (Dolbeault operators)

The operators

\[ \partial : \Omega^{r,s} \to \Omega^{r+1,s}, \quad \bar{\partial} : \Omega^{r,s} \to \Omega^{r,s+1} \]  

(A.12)

from the decomposition \(d = \partial + \bar{\partial}\) are called Dolbeault operators. A \(p\)-form \(\omega \in \Omega^{p,0}\) for which \(\bar{\partial}\omega = 0\) is called holomorphic.

We are now ready to give a criterion to determine whether an almost complex structure actually is a complex structure.

**Theorem A.6**

An almost complex manifold \((M, J)\) admits a complex structure if and only if the Nijenhuis tensor field \(N : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)\) defined by


(A.13)
vanishes \cite{38}.

In local coordinates (A.2), the components of $N$ are

$$N_{\mu \nu} = \mathcal{J}_{\nu}^{\sigma} \partial_{\sigma} \mathcal{J}_{\mu}^{\sigma} - \mathcal{J}_{\mu}^{\sigma} \partial_{\sigma} \mathcal{J}_{\nu}^{\mu}. \quad (A.14)$$

If this condition is fulfilled, one can define complex coordinates in every patch in terms of which the almost complex structure $\mathcal{J}$ can be written as

$$\mathcal{J} = i dz^j \otimes \frac{\partial}{\partial z^j} - i d\bar{z}^j \otimes \frac{\partial}{\partial \bar{z}^j}. \quad (A.15)$$

**Definition A.7 (Hermitian metric)**

A Riemannian metric $g$ on a complex manifold $(M, \mathcal{J})$ is called *Hermitian*, if it satisfies

$$g(X, Y) = g(\mathcal{J}X, \mathcal{J}Y) \quad (A.16)$$

for all vector fields $X, Y$ on $M$. In that case, $(M, g)$ is called *Hermitian manifold*.

It is easy to show that every complex manifold admits a Hermitian metric \cite{39}. In local coordinates, $g = g_{\mu \nu} dx^\mu \otimes dx^\nu$ and condition (A.16) translates as

$$g_{\mu \nu} = \mathcal{J}_{\mu}^{\alpha} \mathcal{J}_{\nu}^{\beta} g_{\alpha \beta}. \quad (A.17)$$

We note the following property \cite{38}: The components of a Hermitian metric with respect to a complex basis,

$$g_{ij}(p) = g_p \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j} \right), \quad g_{ij}(p) = g_p \left( \frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial z^j} \right), \ldots, \quad (A.18)$$

satisfy

$$g_{ij} = 0 = g_{\bar{i} \bar{j}}. \quad (A.19)$$

Thus, one can write

$$g = g_{ij} \left( dz^i \otimes d\bar{z}^j + d\bar{z}^j \otimes dz^i \right). \quad (A.20)$$
A.2 Kähler geometry

Definition A.8 (Kähler form)
On a Hermitian manifold \((M, g, J)\), the tensor field \(J\) defined by

\[
J(X, Y) := g(JX, Y) \quad \text{for all } X, Y \in \Gamma(TM)
\]  

(A.21)

is antisymmetric\(^{25}\) and thus a two-form. Extending \(J\) from \(\Gamma(TM)\) to \(\Gamma(TM)^{\mathbb{C}}\), it becomes a two-form of type \((1, 1)\) and is called Kähler form of the Hermitian metric.

It’s easy to see that the Kähler form is invariant under \(J\), i.e. \(J(JX, JY) = J(X, Y)\) for all vector fields \(X, Y\). If we write \(J\) in a complex basis,

\[
J = J_{ij} dz^i \otimes dz^j + J_{ij} d\bar{z}^i \otimes d\bar{z}^j + \cdots
\]  

(A.22)

we find that due to (A.19) the pure index components vanish and we can write

\[
J = i g_{ij} dz^i \wedge d\bar{z}^j.
\]  

(A.24)

Note that the Kähler form is real, \(\bar{J} = J\). The Kähler form of a Hermitian manifold \(M\) with complex dimension \(m\) allows us to define a \(2m\)-form

\[
J \wedge \cdots \wedge J.
\]  

(A.25)

Writing \(z^i = x^i + iy^i\) and using \(\sqrt{g} = 2^m \det g_{ij}\), we find

\[
\frac{1}{m!} \int_Y J \wedge \cdots \wedge J = \int_Y \sqrt{g} dx^1 \wedge \cdots \wedge dx^m \wedge dy^1 \wedge \cdots \wedge dy^m = \text{Vol}(Y)
\]  

(A.26)

Hence, (A.25) serves as a volume form on the manifold.

Definition A.9 (Kähler manifold)
A Hermitian manifold \((M, g)\) with closed Kähler form \(dJ = 0\) is called Kähler manifold and its metric Kähler metric.

\(^{25}\) We have \(g(JX, Y) = g(J^2X, JY) = -g(X, JY) = -g(JY, X)\).
In local complex coordinates,
\[ \mathrm{d}J = (\partial + \bar{\partial})ig_{ij}dz^i \wedge d\bar{z}^j = 0 \]
from which we conclude
\[ \frac{\partial g_{ij}}{\partial \bar{z}^l} = \frac{\partial g_{il}}{\partial z^j}, \quad \frac{\partial g_{ij}}{\partial \bar{\bar{z}}^l} = \frac{\partial g_{ik}}{\partial \bar{z}^j}. \]
Thus, we can write \( J \) in terms of a so-called Kähler potential \( K \):
\[ g_{ij} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^j}, \quad J = \partial \bar{\partial} K. \] (A.27)

Note that the metric can be expressed this way in a given patch \( U_i \). Given two charts \( (U_i, \varphi_i) \) and \( (U_j, \varphi_j) \) with \( U_i \cap U_j \neq \emptyset \) and coordinates \( z = \varphi_i(p), w = \varphi_j(p) \), the Kähler potentials \( K_i \) and \( K_j \) do in general not coincide but are related by a Kähler transformation
\[ K_j(w, \bar{w}) = K_i(z, \bar{z}) + f_{ij}(z) + g_{ij}(\bar{z}) \] (A.28)
with holomorphic (antiholomorphic) functions \( f \) and \( g \) [38].

With help of the Dolbeault operator \( \bar{\partial} \), we can generalize de-Rham cohomology and define

**Definition A.10** (Dolbeault cohomology group)
For a complex manifold \( M \), we call the quotient space of \( \bar{\partial} \)-closed modulo \( \bar{\partial} \)-exact \((r, s)\)-forms,
\[ H^{r,s}(M) := \frac{\ker(\bar{\partial} : \Omega^r(M) \to \Omega^{r+1}(M))}{\text{im}(\bar{\partial} : \Omega^{r-1}(M) \to \Omega^r(M))} \] (A.29)
the \( X(r, s) \)-th Dolbeault cohomology group. Its complex dimension
\[ h^{r,s} := \dim_{\mathbb{C}} H^{r,s}(M) \] (A.30)
is called Hodge number.

Note that since the Kähler form \( J \) is closed, \( J \in H^{r,s}(M) \). The corresponding equivalence class \([J]\) is called Kähler class.

We further define the adjoint operators \( \partial^\dagger := - \ast \bar{\partial} \ast \) and \( \bar{\partial}^\dagger := - \ast \partial \ast \) with the Hodge-\ast operator and the Laplacians \( \Delta_{\theta} := (\partial + \partial^\dagger)^2 \) and \( \Delta_{\bar{\theta}} := (\bar{\partial} + \bar{\partial}^\dagger)^2 \).
A.3 Calabi-Yau manifolds

We quote the following properties of a Kähler manifold with complex dimension \( m \) [38]:

i) The Hodge numbers \( h^{r,s} \) are related to the Betti numbers \( b^p \) via

\[
b^p = \sum_{r+s=p} h^{r,s},
\]  
(A.31)

ii) they are symmetric, \( h^{r,s} = h^{s,r} \) and \( h^{r,s} = h^{m-r,s-r} \) and

iii) the Laplacians defined above coincide with \( \Delta = (d + d^\dagger)^2 \),

\[
\Delta = 2\Delta_\bar{\partial} = 2\Delta_\partial.
\]  
(A.32)

Definition A.11 (Harmonic form)
A form satisfying \( \Delta_\bar{\partial} = 0 \) is called harmonic and we denote the set of harmonic \((r,s)\)-forms by \( \text{Harm}^{r,s}(M) \).

The Kähler form is harmonic. Forms of this type are of great importance to us since massless forms in the type II supergravity actions are expanded in terms of harmonic forms on the compactification manifold. This is possible due to Hodge’s theorem: For \( \Omega^{r,s}(M) \) there is a unique decomposition

\[
\Omega^{r,s}(M) = \bar{\partial}\Omega^{r,s-1}(M) \oplus \bar{\partial}\Omega^{r,s+1}(M) \oplus \text{Harm}^{r,s}(M).
\]  
(A.33)

From this follows in particular that

\[
\text{Harm}^{r,s}(M) \cong H^{r,s}(M).
\]  
(A.34)

A.3 Calabi-Yau manifolds

There are many different ways to define what a Calabi-Yau is. We cite from [40]:

Theorem A.12
Let \( (Y,J,g) \) be a compact Kähler manifold with complex dimension \( \dim\mathbb{C}(Y) = n \). The following statements are equivalent:

i) \( Y \) is Ricci-flat, that is it has vanishing Ricci-form \( \mathcal{R} = 0 \).

ii) It admits a globally defined and nowhere vanishing holomorphic \( n \)-form.

iii) It has holonomy \( \text{Hol}(g) \subset SU(n) \).

iv) Its first Chern class vanishes.
**Definition A.13** (Calabi-Yau n-fold)
A compact Kähler $n$-fold with one of the properties listed in A.12 is called *Calabi-Yau* manifold.

From now on, we denote the coordinates on a Calabi-Yau $Y$ by \( \{y^i, \bar{y}^j\} \) and only consider those of complex dimension \( \text{dim}_C Y = 3 \). For a Calabi-Yau threefold, the Hodge numbers are depicted in the following figure:

In particular, there is only one \((3,0)\)-form which we denote by $\Omega$ (with $\bar{\Omega}$ the conjugate \((0,3)\)-form), while there are no one- or five-forms. It is worth noticing that since $h^{3,3} = 1$, the \((3,3)\)-form $\Omega \wedge \bar{\Omega}$ must be proportional to $J \wedge J \wedge J$ and thus to the volume form.

As presented in the main text, for a Calabi-Yau $Y$ the cohomology groups $H^{1,1}(Y)$ and $H^{2,1}(Y)$ are associated with Kähler and complex structure moduli respectively. Thus, we introduce

i) a basis $\{\omega_A\}$ for $H^{1,1}(Y)$ with $A = 1, \ldots, h^{1,1}$ as well as a dual basis $\{\tilde{\omega}^A\}$ for $H^{2,2}(Y)$ normalized such that

$$
\frac{1}{V_0} \int_Y \omega_A \wedge \tilde{\omega}_B = \delta^A_B,
$$

where we introduced a six-dimensional fixed Calabi-Yau references volume $V_0$.

ii) a basis $\{\eta_a\}$ for $H^{2,1}(Y)$ and $\{\bar{\eta}_a\}$ for $H^{1,2}(Y)$ with $a = 1, \ldots, h^{2,1}$.

\[
\begin{array}{ccccccc}
& h^{0,0} & h^{1,0} & h^{2,0} & h^{3,0} & h^{3,1} & h^{3,2} & h^{3,3} \\
\text{h}^{1,0} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{h}^{2,0} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\text{h}^{3,0} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\text{h}^{3,1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\text{h}^{3,2} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\text{h}^{3,3} & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{array}
\]

**Figure 14:** Hodge Diamond of a Calabi-Yau threefold.
A.4 Some integrals on Calabi-Yau threefolds

We collect a couple of integrals used in the thesis for the compactification of Type II supergravity. First, we need to know how to deal with $\alpha^a \wedge \ast \beta_b$ and similar terms. The real three-forms $\alpha^a$ and $\beta^b$ were defined to be orthogonal as stated in (3.25). Since their Hodge duals are again three-forms, we can expand

\[ \ast \alpha^a = A^b \alpha_b + B_{ab} \beta^b, \quad \ast \beta^a = C^\delta \alpha^\delta + D_{\delta b} \beta^b \]  

(A.36)

in terms of some coefficient matrices $A, B, C$ and $D$. Thus,

\[ \frac{1}{V_0} \int_Y \alpha^a \wedge \ast \alpha_b = \frac{1}{V_0} \int_Y \alpha^a \wedge (B^c_b \beta^c) = B_{ab} \delta^a_b = B_{ab} \]  

(A.37)

and analogously

\[ \frac{1}{V_0} \int_Y \beta^a \wedge \ast \beta^b = -C^b_a = -C_{ab}, \quad \frac{1}{V_0} \int_Y \alpha^a \wedge \ast \beta^b = D^b_a = -A_{ab}. \]  

(A.38)

These matrices can be written in terms of a matrix $M$ via

\[ A = (\text{Re} \mathcal{M})(\text{Im} \mathcal{M})^{-1}, \quad B = -(\text{Im} \mathcal{M}) - (\text{Re} \mathcal{M})(\text{Im} \mathcal{M})^{-1}(\text{Re} \mathcal{M}), \quad C = (\text{Im} \mathcal{M})^{-1}, \]  

(A.39)

which again is determined by the moduli. This is derived in C.3 and we note at this point:

\[ \frac{1}{V_0} \int_Y \alpha^a \wedge \ast \alpha_b = -[\text{Im} \mathcal{M} + (\text{Re} \mathcal{M})(\text{Im} \mathcal{M})^{-1}(\text{Re} \mathcal{M})]_{ab}, \]

\[ \frac{1}{V_0} \int_Y \beta^a \wedge \ast \beta^b = -[(\text{Im} \mathcal{M})^{-1}]^\delta_a \delta^b, \]

\[ \frac{1}{V_0} \int_Y \alpha^a \wedge \ast \beta^b = -[(\text{Re} \mathcal{M})(\text{Im} \mathcal{M})^{-1}]^b_a, \]

\[ \frac{1}{V_0} \int_Y \alpha^a \wedge \beta^b = \delta^b_a \]  

(A.40)

If we integrate one of the four-dimensional forms $\Lambda$ over the internal Calabi-Yau $Y$, we just get the form times the volume,

\[ \int_Y \Lambda \wedge \ast \Lambda = \Lambda \wedge \ast \Lambda \int_Y \ast 1 = \Lambda \wedge \ast \Lambda \int_Y \ast = \frac{1}{6} \mathcal{K} \Lambda \wedge \ast \Lambda. \]  

(A.41)
B Type II Supergravity

In the thesis, we are dealing with the low energy effective actions of Type IIA and IIB string theory, namely Type IIA and Type IIB supergravity (SUGRA). This appendix reviews some facts about Type II supergravity that are important for the thesis.

In supersymmetry (SUSY), the Poincaré and internal symmetries of a quantum theory are extended to include $N$ charges which are spinors. The Poincaré and SUSY charges form a so-called superalgebra that necessarily includes bosonic and fermionic elements. Supergravity is supersymmetry where the SUSY parameters $\varepsilon = \varepsilon(x)$ are local. The name hints at the fact that local supersymmetry necessarily includes gravity and conversely, a consistent supersymmetric theory of gravity must be local. This is standard textbook material and a thorough treatment can be found in the literature. See [21, 22], [19] or [41] for general string theory and [42], [23] or the appendix of [22] for supersymmetry and supergravity. Both here and in the rest of the thesis, a basic knowledge of these topics is presumed.

Supergravity restricts the number of spacetime dimensions in which it can be consistently formulated to eleven as a higher-dimensional theory leads to more than eight gravitinos upon toroidal compactification to four dimensions. These can only be embedded in a representation that also has spins $\geq 5/2$ for which - as is well-known - no consistent interactions exist [23].

B.1 Type IIA SUGRA from compactification of 11-dim. SUGRA

We begin with SUGRA in the largest allowed dimension $d = 11$. This is not only the low-energy effective action of M-theory but we will construct Type IIA supergravity by compactifying $d = 11$-SUGRA on a circle. It contains:

- The graviton $g_{MN}$ transforming in the symmetric traceless of $SO(d - 2)$ which has dimension $d(d - 3)/2 = 44$.
- Its superpartner, the gravitino $\psi_M$: A Majorana spinor in the vector-spinor representation of $SO(d - 2)$ which has dimension $(d - 3)2^{(d-2)/2} = 128$.
- A three-form $C_3$ that accounts for the remaining $128 - 44 = 84 = \binom{d-2}{3}$ bosonic states.

The unique action keeping only the bosonic fields is

$$S^{(11)} = \frac{1}{2\kappa_{11}^2} \int *R - \frac{1}{2\kappa_{11}^2} \int F_4 \wedge *F_4 - \frac{1}{6\kappa_{11}^2} \int F_4 \wedge F_4 \wedge C_3$$  (B.1)
with \( F_4 = dC_3 \) and the gravitational coupling \( \kappa_{11} \). In \( d \) dimensions, the latter relates to the \( d \)-dimensional Newton’s constant \( G_d \) via

\[
\kappa_d^2 = 8\pi G_d. \tag{B.2}
\]

As mentioned before, Type IIA SUGRA is obtained upon dimensional reduction of \( S^{(11)} \), i.e. compactification on \( S^1 \) keeping only the massless modes\(^{26}\). In the following, we will sketch the usual Kaluza-Klein procedure as presented for instance in \([21]\).

We take one of the spatial dimensions and call it \( y \), demanding that \( y \cong y + 2\pi R \), where \( R \) is the radius of the compactification circle. This corresponds to \( \mathbb{R}^{1,10} \to \mathbb{R}^{1,9} \times S^1 \) and gives rise to a scalar \( \sigma \) and a gauge field \( A_1 \) from the lower-dimensional perspective. To see this, consider the general ansatz

\[
\hat{G}_{MN}(x,y)dx^Mdx^N = G_{\mu\nu}(x) + e^{2\sigma(x)}(dy + A_\mu(x)dx^\nu)^2, \quad \mu, \nu = 0, ..., 9, \tag{B.3}
\]

or

\[
\hat{G}_{MN} = \left( \begin{array}{cc} G_{\mu\nu} + e^{2\sigma}A_\mu A_\nu & e^{2\sigma}A_\mu \\
 e^{2\sigma}A_\nu & e^{2\sigma} \end{array} \right), \tag{B.4}
\]

where the hats denote \( 11 \)-dim. quantities. Likewise, the three-form \( \hat{C}_3 \) splits into a three-form \( C_3 \) and a two-form \( B_2 \). The precise procedure is discussed in section 3 and we won’t carry out the explicit compactification but rather quote the resulting ten-dimensional action from \([22]\):

\[
S^{(10)} = \frac{1}{2\kappa_{10}^2} \int \left( e^{\sigma} \ast R - \frac{1}{2} e^{3\sigma} F_2 \wedge \ast F_2 \right) - \frac{1}{4\kappa_{10}^2} \int \left( e^{-\sigma} H_3 \wedge \ast H_3 + e^\sigma F_4 \wedge \ast F_4 \right) - \frac{1}{4\kappa_{10}^2} \int B_2 \wedge F_4, \tag{B.5}
\]

where we defined the ten-dimensional coupling \( \kappa^2 = 2\pi R\kappa_{10}^2 \) and

\[
H_3 = dB_2, \quad F_4 = dC_3 - A_1 \wedge H_3. \tag{B.6}
\]

To make contact with string theory, we perform a Weyl-rescaling \([35, 37]\)

\[
G_{\mu\nu} \to \Omega^{-2} G_{\mu\nu} \tag{B.7}
\]

\(^{26}\)This is due to the fact that M-theory compactified on a circle with radius \( R \) is corresponding to Type IIA string theory with string coupling \( g_s = R/\sqrt{\alpha'} \).
under which
\[
G^{\mu
u} \rightarrow \Omega^2 G^{\mu
u}, \quad \sqrt{-G} \rightarrow \Omega^{-d}\sqrt{-G},
\]

\[
\int d^d\sqrt{-G}R \rightarrow \int d^d\sqrt{-G}\Omega^{2-d} (R + (d - 1)(d - 2)\Omega^{-2}\partial_\mu\Omega\partial^\mu\Omega). \quad (B.8)
\]

Note that wedge products of the form \( C_p \wedge \ast C_p \) contain \( \sqrt{-G} \) and \( p \) times the metric, such that
\[
C_p \wedge \ast C_p \rightarrow \Omega^{2p-d}C_p \wedge \ast C_p. \quad (B.9)
\]

With \( \Omega = e^{ \frac{2}{3} \phi} \) and \( d = 10 \), the action is
\[
S^{(10)} = \frac{1}{2\kappa_{10}^2} \int \left[ e^{-3\sigma} (\ast R + 9 \cdot 2d\sigma \wedge \ast d\sigma) - \frac{1}{2} F_2 \wedge \ast F_2 \right]
- \frac{1}{4\kappa_{10}^2} \int (e^{-3\sigma} H_3 \wedge \ast H_3 + F_4 \wedge \ast F_4)
- \frac{1}{4\kappa_{10}^2} \int B_2 \wedge F_4 \wedge F_4. \quad (B.10)
\]

Defining \( \sigma = \frac{2}{3}\phi \) and rearranging the terms, we finally get
\[
S^{(10)}_{\text{IIA}} = \frac{1}{2\kappa_{10}^2} \int e^{-2\phi} \left( \ast R + 4d\phi \wedge \ast d\phi - \frac{1}{2} H_3 \wedge \ast H_3 \right)
- \frac{1}{4\kappa_{10}^2} \int (F_2 \wedge \ast F_2 + F_4 \wedge \ast F_4) - \frac{1}{4\kappa_{10}^2} \int B_2 \wedge F_4 \wedge F_4 \quad (B.11)
\]

which has the structure
\[
S^{(10)}_{\text{IIA}} = S_{\text{NS}} + S_{\text{R}}^{\text{IIA}} + S_{\text{CS}}^{\text{IIA}} \quad (B.12)
\]

with fields in the NS-NS and R-R sector of Type IIA string theory and a Chern-Simons term.

\section*{B.2 Type IIB SUGRA}

Since we only consider massless fields, the difference between the field content of Type IIA and Type IIB string theory lies in the R-R sector where the former contains the odd \( p \)-form gauge fields \( A_1 \) and \( C_3 \) while the latter has the even fields...
IIB thus suggests to define similarly
\[ S_{(10)\text{IIB}} = S_{\text{NS}} + S_{\text{IR}} + S_{\text{CS}}. \] (B.13)

Explicitly,
\[ S_{(10)\text{IIB}} = \frac{1}{2\kappa_{10}^2} \int e^{-2\phi} \left( *R + 4d\phi \wedge *d\phi - \frac{1}{2} H_3 \wedge *H_3 \right) \]
\[ - \frac{1}{4\kappa_{10}^2} \int (F_1 \wedge *F_1 + F_3 \wedge *F_3 + F_5 \wedge *F_5) - \frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3 \] (B.14)

with
\[ H_3 = dB_2, \]
\[ F_1 = dC_0, \]
\[ F_3 = dC_2 - C_0 H_3, \]
\[ F_5 = dC_4 + B_2 \wedge dC_2. \] (B.15)

There is an obstacle though, since self-duality \( F_5 = *F_5 \) is not implied by (B.14) and thus must be imposed by an additional constraint on the solution. As a consequence, the above action does not have the same number of bosonic and fermionic degrees of freedom in thus is not supersymmetric. The equations of motion, though, are supersymmetric if the constraint is imposed.

B.3 \( \mathcal{N} = 2 \) supergravity in \( d = 4 \) and special geometry

As discussed in section 3, the four-dimensional theory resulting from compactification of Type II string theory on a Calabi-Yau manifold has \( \mathcal{N} = 2 \) supersymmetry. Hence, the actions obtained from the Type II supergravities are \( \mathcal{N} = 2 \) supergravities in four dimensions. We briefly review some of their features needed in the thesis - a comprehensive treatment can be found e.g. in [23]

B.3.1 Multiplets of \( \mathcal{N} = 2 \) supergravity

An extensive treatment on this can be found in [23]. The (massless) field content of \( \mathcal{N} = 2 \) SUGRA in \( d = 4 \) is given by \( n_V \) one-forms \( V^A \) and complex scalars \( z^A \), the metric \( g \) and a one-form \( V^0 \) called the graviphoton as well as \( 2n_H \) additional complex scalars \( t^a, \xi^a \). These form three multiplets that are listed in table 5. The \( \sigma \)-model describing their self-interactions factorizes in a product of a special Kähler manifold for the scalars \( z^A \) in the gauge multiplets and a quaternionic manifold for
B.3 $\mathcal{N} = 2$ supergravity in $d = 4$ and special geometry

<table>
<thead>
<tr>
<th>Multiplet</th>
<th>(Massless) field content</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gravity multiplet</td>
<td>$(g_{\mu\nu}, A^0)$</td>
<td>1</td>
</tr>
<tr>
<td>Gauge multiplets</td>
<td>$(V^A, z^A)$</td>
<td>$n_V$</td>
</tr>
<tr>
<td>Hypermultiplets</td>
<td>$(t^a, \xi^a)$</td>
<td>$n_H$</td>
</tr>
</tbody>
</table>

Table 5: $\mathcal{N} = 2$ supergravity multiplets in $d = 4$.

the scalars in the hypermultiplets [30]. It is the former scalars that parameterize the matrices $I$ and $R$ in 4.1. Since there are $n_V + 1$ vectors, the special Kähler manifold is projective and we can use $n_v + 1$ holomorphic sections $Z^\hat{A}(z^A)$ as projective coordinates on the scalar manifold as well as $n_V + 1$ dual vector field $\mathcal{F}_{\hat{A}}$ such that the sections $(Z^\hat{A}, \mathcal{F}_{\hat{A}})$ form symplectic vectors. We will now elaborate on this in more detail. First, we will discuss special geometry for global SUSY since it is a little easier before turning to SUGRA. For further discussion on this topic see [37, 43] and especially [23].

B.3.2 Rigid special Kähler manifolds

**Definition B.1** (Rigid special Kähler geometry)

Let $(M, g)$ be a Kähler manifold with dim$_\mathbb{C} M = n$ on which complex coordinates $z^a$ and $2n$ fields $X^A(z)$, $F_A(z)$ are defined where the latter transform as a vector of the symplectic group $Sp(2n, \mathbb{R})$. Writing $V = (X^A, \mathcal{F}_A)$, we can define an inner product

$$\langle V, W \rangle = V^T \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} W$$

which obviously is invariant under symplectic transformations. We call $M$ a rigid special Kähler manifold if

$$g_{\hat{i}\hat{j}} = i \langle \partial_i V, \partial_j \bar{V} \rangle, \quad \langle \partial_i V, \partial_j V \rangle = 0.$$  \hspace{1cm} (B.17)

Note that $i \langle \partial_i V, \partial_j \bar{V} \rangle = \partial_i \partial_j i \langle V, \bar{V} \rangle$, i.e. the symplectic invariant $i \langle V, \bar{V} \rangle$ serves as a Kähler potential for the metric. Since both indices $i$ and $I$ range from 1, ..., $n$, the expression

$$\partial_i X^I = \frac{\partial X^I}{\partial z^i}$$

\hspace{1cm} (B.18)
is an $n \times n$ matrix. If it is invertible - which is the case if $g$ is positive definite [23] - then we can choose the $X^I$ as new coordinates and multiply the second equation in (B.17) with the inverse $\partial z^j/\partial X^K$ to get

$$\frac{\partial F_K}{\partial z^j} = \frac{\partial F_I}{\partial X^K} \frac{\partial X^I}{\partial z^j}. \quad (B.19)$$

This is the condition for $F_I$ to possess a prepotential $F$ which is a holomorphic function of the coordinates $X$ with

$$F_I(X) = \frac{\partial F(X)}{\partial X^I}. \quad (B.20)$$

Since we are eventually interested in $\mathcal{N} = 2, d = 4$ SUGRA obtained from Type II string theory, we now turn to so-called projective special Kähler geometry. Henceforth, we will drop the word “projective”.

### B.3.3 Special Kähler manifolds

Special geometry in its symplectic formulation is suited to describe the moduli spaces of Calabi-Yau threefolds. In (3.50), the fields $Z^\hat{a}$ and $F_{\hat{a}}$, which are defined as periods of the holomorphic three-form$^{27}$,

$$Z^{\hat{a}} = \int_{A_\hat{a}} \Omega, \quad F_{\hat{a}} = \int_{B^{\hat{a}}} \Omega, \quad (B.21)$$

form a symplectic vector

$$v := \begin{pmatrix} Z^{\hat{a}}(z) \\ F_{\hat{a}}(z) \end{pmatrix} \quad (B.22)$$

with $\hat{a} = 0, 1, ..., h^{2,1}$. Note that unlike the case of rigid special geometry, this is one degree of freedom more than the number of coordinates $z^a$. However, since a rescaling of $\Omega$ amounts to a rescaling of the periods $Z^{\hat{a}}$ while the holomorphic three-form is defined only up to a complex rescaling, the $Z^{\hat{a}}$ are projective,

$$(Z^0, Z^1, ..., ) \cong (\lambda Z^0, \lambda Z^1, ...). \quad (B.23)$$

\[27\] Note that unlike the main text, we are not careful about a Calabi-Yau reference volume factor $Y_0$ in this appendix. It can always be reinstalled by dimensional analysis, though.
Defining $h^{2,1}$ inhomogeneous coordinates\(^{28}\) $Z^\hat{a}/Z^0$ - provided the matrix

$$\partial_a \left( \frac{Z^b}{Z^0} \right)$$

is invertible - we have the right number to chose

$$z^a = \frac{Z^a}{Z_0}$$

or $Z^\hat{a} = (1, z^a)$. Hence, we can write the periods $F_\hat{a}$ as functions\(^{29}\) of the coordinates $Z$. We arrive at a condition similar to the second equation of (B.17) from

$$\int \partial_\hat{a} \Omega \wedge \Omega = \int (\alpha_\hat{a} - \partial_a F_\hat{b} \beta^b) \wedge (Z^\hat{a} \alpha_\hat{a} - F_\hat{a} \beta^\hat{a})$$

$$= - F_\hat{a} + (\partial_a F_\hat{b}) Z^b,$$

(B.26)

where $\partial_\hat{a} = \partial/\partial Z^\hat{a}$, since this integral vanishes. This follows directly from the expansion

$$\partial_a \Omega = k_a \Omega + i \eta_a$$

(B.27)

derived in (C.1.1) and we conclude

$$F_\hat{a} = (\partial_a F_\hat{b}) Z^b = \frac{1}{2} \partial_a \left( F_\hat{b} Z^\hat{b} \right).$$

(B.28)

Thus, similar to (B.20), the periods $F_I$ can be written in terms of a prepotential via

$$F_\hat{a} = \frac{\partial F(Z)}{\partial Z^\hat{a}}, \text{ where } F(Z) = \frac{1}{2} F_\hat{a} Z^a.$$ 

(B.29)

Unlike the rigid case, the symplectic product $i\langle v, \bar{v} \rangle$ does not serve as a Kähler potential. Rather, we define

$$e^{-K} := -i\langle v, \bar{v} \rangle$$

(B.30)

and we will see in a minute, that $K$ is a Kähler potential for the metric on the complex structure moduli space. As mentioned several times, the holomorphic

\(^{28}\)In a chart where $Z^0 = 0$, we can chose a $Z^b \neq 0$, since the $Z^\hat{a}$ do not simultaneously vanish, and rename the indices.

\(^{29}\)By a small abuse of notation, we use the same name for $F_\hat{a}(z)$ and $F_\hat{a}(Z)$ again.
three-form is not uniquely defined but we can consider redefinitions

\[ \Omega \to e^{f(Z)} \Omega \]  

that should not change the physics. We see that under (B.31), the quantity \( K \) transform as

\[ K \to K - f(Z) - \bar{f}(\bar{Z}) \]  

which is a Kähler transformation. Now, we are almost in place to write down an expression for the Kähler metric similar to the rigid case (B.17). First, we define the vectors

\[ V := e^{1/2K}v \]  

as well as \textit{Kähler covariant derivatives}

\[
\nabla_a Z^b := \partial_a Z^b + (\partial_a K)Z^b, \\
\nabla_{\bar{a}} \bar{Z}^b := \partial_{\bar{a}} \bar{Z}^b + (\partial_{\bar{a}} K)\bar{Z}^b. 
\]

Note that these transform as

\[ \nabla_a Z^b \to \nabla_a Z^b e^{-f}, \quad \nabla_{\bar{a}} \bar{Z}^b \to \nabla_{\bar{a}} \bar{Z}^b e^{-\bar{f}} \]  

under a combined Kähler transformation

\[ Z^\hat{a} \to Z^\hat{a} e^{-f}, \quad \bar{Z}^\hat{a} \to \bar{Z}^\hat{a} e^{-\bar{f}}, \quad K \to K + f + \bar{f}. \]

Then, the expression

\[ i \langle \nabla_a V, \nabla_{\bar{b}} \bar{V} \rangle \]  

where \( \nabla_a V = e^K \nabla v \) is Kähler and symplectic covariant. Explicitly,

\[
i \langle \nabla_a V, \nabla_{\bar{b}} \bar{V} \rangle = ie^K (\partial_a \partial_{\bar{b}} \langle v, \bar{v} \rangle + \partial_a \langle v, \bar{v} \rangle \partial_{\bar{b}} K + \partial_a K \partial_{\bar{b}} \langle v, \bar{v} \rangle + \partial_a K \partial_{\bar{b}} K \langle v, \bar{v} \rangle) \\
= (\partial_a \partial_{\bar{b}} K + \partial_a K \partial_{\bar{b}} K) + (\partial_a K \partial_{\bar{b}} K) + (\partial_a K \partial_{\bar{b}} K) + (\partial_a K \partial_{\bar{b}} K) \\
eq \partial_a \partial_{\bar{b}} K, 
\]

which means that it is the Kähler metric corresponding to the Kähler potential \( K \). In the discussion of the complex structure moduli space in section 3.5 we see that this is the Kähler metric \( G^a_{ab} \).
C Calculations

C.1 Compactification

C.1.1 Expansion of $\partial z^a \Omega$

We derive eq. (3.54). First, we show that $\partial z^a \Omega \in H^{3,0} + H^{2,1}$:

$$\partial z^a \Omega = \frac{1}{3!} (\partial z^a \Omega_{ijk}) dy^i \wedge dy^j \wedge dy^k + \frac{1}{2} \Omega_{ijk} \partial z^a (dy^i) \wedge dy^j \wedge dy^k. \tag{C.1}$$

The first part is a $(3,0)$-form, while the second term is the wedge product of the derivative $\partial z^a dy^v$ and a $(2,0)$-form. We will take a closer look at this derivative. Expanding

$$y^i (z^a + \delta z^a) = y^i (z^a) + \Lambda^i_a \delta z^a, \tag{C.2}$$

we find

$$\partial z^a (dy^i) = d\Lambda^i_a = \frac{\partial \Lambda^i_a}{\partial y^j} dy^j + \frac{\partial \Lambda^i_a}{\partial \bar{y}^j} d\bar{y}^j. \tag{C.3}$$

Thus, $\partial z^a dy^v$ is composed of a $(1,0)$-form and a $(0,1)$-form, i.e. $\partial z^a \Omega \in H^{3,0} + H^{2,1}$. We calculate the $(2,1)$-part

$$\frac{1}{2} \Omega_{ijk} \frac{\partial \Lambda^i_a}{\partial y^j} dy^i \wedge dy^j \wedge dy^k \tag{C.4}$$

by differentiating the Calabi-Yau metric:

$$0 = \partial z^a (2g_{ij} dy^i d\bar{y}^j) = 2 \frac{\partial g_{ij}}{\partial z^a} dy^i d\bar{y}^j + 4 g_{ij} \frac{\partial \Lambda^k_a}{\partial y^j} dy^i d\bar{y}^j \tag{C.5}$$

and find

$$g^{ik} g_{kj} \frac{\partial g_{ij}}{\partial z^a} = \frac{\partial g_{ij}}{\partial z^a} = -2 g_{kj} \partial_i \Lambda^k_a. \tag{C.6}$$

From (3.37), we have

$$\frac{\partial g_{ij}}{\partial z^a} = - \frac{i}{\| \Omega \|^2} (\eta_a)_{ikl} \bar{\Omega}^{kl}_j \tag{C.7}$$
with $\partial_i := \partial/\partial y^i$, that is,

$$\partial_i \Lambda^k_a = \frac{i}{2\|\Omega\|^2} (\eta_a)_{im} \bar{\Omega}^m_{nk}.$$  \hspace{1cm} (C.8)

Thus, we confirm that (C.4) is given by $\eta_a$, i.e.

$$\frac{\partial}{\partial z^a} \Omega = k_a \Omega + i \eta_a.$$  \hspace{1cm} (C.9)

### C.1.2 Kähler moduli metric

We derive the metric (3.69) from the Kähler moduli part of (3.44):

$$2G_{AB} t^A t^B = -\frac{1}{2V} \int d^6 x \sqrt{g} g^{ik} g^{jl} \left( \delta g_{il} \delta g_{jk} - \delta B_{il} \delta B_{jk} \right)$$

$$= -\frac{1}{2V} \int d^6 x \sqrt{g} g^{ik} g^{jl} \left[ (-i)^2 v^A_{i} (\omega_A)_j v^B_k (\omega_B)_l - b^A_{i} (\omega_A)_j b^B_{k} (\omega_B)_l \right]$$

$$= \frac{1}{2V} \int d^6 x \sqrt{g} g^{ik} g^{jl} (v^A_{i} v^B_j + b^A_{i} b^B_{j}) (\omega_A)_i (\omega_B)_j$$

$$= \frac{1}{2V} \int \omega_A \wedge * \omega_B t^A t^B,$$  \hspace{1cm} (C.10)

that is,

$$G_{AB} = \frac{1}{4V} \int \omega_A \wedge * \omega_B.$$  \hspace{1cm} (C.11)

### C.1.3 Kähler moduli space prepotential

It is shown that the cubic prepotential indeed is a prepotential for the Kähler moduli:

$$i (X^A \bar{F}_A - X^A \bar{F}_A) = -\frac{i}{3!} K_{ABC} \left( -\bar{X}_0 X^A X^B X^C \frac{X_0^2}{X_0^2} + 3 \bar{X}^A X^B X^C \frac{X_0}{X_0} - c.c. \right)$$

$$= \frac{i}{3!} K_{ABC} (t^A t^B t^C + 3 \bar{t}^A t^B t^C - \bar{t}^A t^B t^C - 3t^A \bar{t}^B \bar{t}^C)$$

$$= \frac{i}{3!} K_{ABC} (t^A - \bar{t}^A) (t^B - \bar{t}^B) (t^C - \bar{t}^C)$$

$$= \frac{i}{3!} K_{ABC} (2i)^3 v^A v^B v^C$$

$$= \frac{4}{3} K.$$  \hspace{1cm} (C.12)
C.2 Supersymmetric black holes

C.2.1 Matrix $\mathcal{M}$ in relation for central charge

We show that

$$\frac{1}{2} (p \quad q) \begin{pmatrix} -(\mathcal{I} + R\mathcal{I}^{-1}R) & R\mathcal{I}^{-1} \\ \mathcal{I}^{-1} & -\mathcal{I}^{-1} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = -\frac{1}{2} (q - Np)^T \mathcal{I}^{-1}(q - \tilde{N}p)$$

(C.13)

in order to prove (4.25). After multiplying both sides by $\frac{1}{2}$, the left-hand side reads

$$-p^A(\mathcal{I} + R\mathcal{I}^{-1}R)_{AB}p^B + p^A(R\mathcal{I}^{-1})_A^B q_B + q_A(\mathcal{I}^{-1}R)^A_B p^B - q_A(\mathcal{I}^{-1})^{AB} q_B.$$  

(C.14)

We find the same $q^T \mathcal{I}^{-1} q$-term on both sides and the remaining terms on the right-hand side are

$$-N_{ABC}p^C(\mathcal{I}^{-1})^{AB} q_B = -p^C(R + i\mathcal{I})_{AC}(\mathcal{I}^{-1})^{AB} q_B$$

$$= -p^A(R\mathcal{I}^{-1})_A^B q_B - iq^A q_A,$$

$$-q_A(\mathcal{I}^{-1})^{AB} \tilde{N}_{BC} p^C = -q_A(\mathcal{I}^{-1})^{AB}(R - i\mathcal{I})_{BC} p^C$$

$$= -q_A(\mathcal{I}^{-1}R)^A_B p^B + iq^A q_A,$$

$$N_{ABC}(\mathcal{I}^{-1})^{AB} \tilde{N}_{BD} p^D = p^C(R + i\mathcal{I})_{AC}(\mathcal{I}^{-1})^{AB}(R - i\mathcal{I})_{BD} p^D$$

$$= -p^A(\mathcal{I} + R\mathcal{I}^{-1}R)_{AB} p^B,$$  

(C.15)

which adds up to precisely what we found for the left-hand side.

C.3 Gauge-coupling matrix

The matrices defined by the expansions

$$*\alpha_a = A^b_a \alpha^b_b + B_{ab} \beta^b_b,$$

$$*\beta^a = C^{ab} \alpha^b_b + D^{ab} \beta^b_b$$

(C.16)

can be expressed in terms of the moduli. To do so, notice that the Hodge star acts on a $(3,0)$-form as $*\Omega = -i\Omega$ which lets us write (3.50) as

$$*\Omega = Z^a_b *\alpha^b_a - F^b_b *\beta^b_a$$

$$= Z^a_b(A^b_a \alpha^b_b + B_{ab} \beta^b_b) - F^b_b(C^{bc} \alpha^c_c - A^b_c \beta^c_c)$$

$$= -i\Omega$$

$$= -i(Z^a_b \alpha^b_a - F^b_b \beta^b_a),$$

(C.17)
and by equating coefficients

\[ Z^\hat{a} A_\hat{b} - F_\hat{a} C^\hat{a} = -i Z^\hat{b}, \]
\[ Z^\hat{a} B_\hat{ab} + A_b^\hat{a} = i F_b. \]  

(C.18)

We found the expression (3.62) for \( k_a \) in the expansion of \( \Omega \), which we now calculate explicitly with help of the prepotential:

\[ k_a = \partial_{z^a} \ln \left( \bar{Z}^b F_b - Z^b \bar{F}_b \right) \]
\[ = \frac{\bar{Z}^b F_{ab} - \bar{F}_a}{i \left( \bar{Z}^b F_b - Z^b \bar{F}_b \right)}, \quad F_a = F_{\hat{a}} Z^\hat{b} \]
\[ = \frac{Z^b F_{ab} - \bar{F}_a \bar{Z}^b}{Z^b Z^c F_{bc} - Z^b \bar{Z}^c \bar{F}_{bc}} \]
\[ = \frac{2 \text{Im} F_{ab} \bar{Z}^b}{2 \text{Im} F_{bc} \bar{Z}^b Z^c}. \]  

(C.19)

Thus,

\[ \partial_{z^a} \Omega = \frac{1}{\text{Im} F_{bc} \bar{Z}^b Z^c} \text{Im} F_{ab} \bar{Z}^b \Omega + i \eta_a. \]  

(C.20)

Since also \( \partial_{z^0} \Omega \in H^{3,0} + H^{2,1} \), one can define \( \eta_0 \) via

\[ \partial_{z^0} \Omega = \frac{1}{\text{Im} F_{bc} \bar{Z}^b Z^c} \text{Im} F_{0b} \bar{Z}^b \Omega + i \eta_0. \]  

(C.21)

Using equation (C.18), it follows that

\[ \alpha_a - F_{\hat{a} b} \beta^b = \partial_{z^a} \Omega = \frac{1}{\text{Im} F_{bc} \bar{Z}^b Z^c} \text{Im} F_{ab} \bar{Z}^b \Omega + i \eta_a \]
\[ = \frac{1}{\text{Im} F_{bc} \bar{Z}^b Z^c} \text{Im} F_{ab} \bar{Z}^b (Z^c \alpha_c - F_{c\beta^c}) + i \eta_a, \]  

(C.22)
Likewise, one finds

\[ i \eta_a = \left( \delta_a^\delta - \frac{\text{Im } F_{ab} \bar{Z}^b Z^c}{\text{Im } F_{bd} Z^b Z^d} \right) \alpha_{\hat{a}} - \left( F_{\hat{a} c} - \frac{\text{Im } F_{ab} \bar{Z}^b F_{\hat{c}}}{\text{Im } F_{bd} Z^b Z^d} \right) \beta_{\hat{c}} \]

\[ = * \eta_a \]

\[ = -i \left( \delta_a^\delta - \frac{\text{Im } F_{ab} \bar{Z}^b Z^c}{\text{Im } F_{bd} Z^b Z^d} \right) (A_{\hat{c}}^c \alpha_{\hat{c}} + B_{\hat{c} \hat{c}} \beta_{\hat{c}}) \]

\[ + i \left( F_{\hat{a} c} - \frac{\text{Im } F_{ab} \bar{Z}^b F_{\hat{c}}}{\text{Im } F_{bd} Z^b Z^d} \right) (C_{\hat{c} \hat{c}} \alpha_{\hat{c}} - A_{\hat{c}}^c \beta_{\hat{c}}) \]

(C.23)

from which we find

\[ \left( \delta_a^\delta - \frac{\text{Im } F_{ab} \bar{Z}^b Z^c}{\text{Im } F_{bd} Z^b Z^d} \right) (\delta_c^\delta + i A_{\hat{c}}^c) = i \left( F_{\hat{a} c} - \frac{\text{Im } F_{ab} \bar{Z}^b F_{\hat{c}}}{\text{Im } F_{bd} Z^b Z^d} \right) C_{\hat{c} \hat{c}} \]

\[ i (-A_{\hat{c}}^c Z^c + C_{\hat{c} \hat{c}} F_{\hat{c}}) \frac{\text{Im } F_{ab} \bar{Z}^b}{\text{Im } F_{bd} Z^b Z^d} + \delta_a^\delta - \frac{\text{Im } F_{ab} \bar{Z}^b Z^c}{\text{Im } F_{bd} Z^b Z^d} = -i (A_{\hat{c}}^c - F_{\hat{a} c} C_{\hat{c} \hat{c}}). \] (C.24)

Using (C.18) again, this is

\[ -Z^c \frac{\text{Im } F_{ab} \bar{Z}^b}{\text{Im } F_{bd} Z^b Z^d} + \delta_a^\delta - \frac{\text{Im } F_{ab} \bar{Z}^b Z^c}{\text{Im } F_{bd} Z^b Z^d} = -i (A_{\hat{c}}^c - F_{\hat{a} c} C_{\hat{c} \hat{c}}), \] (C.25)

i.e.

\[ A_{\hat{a}}^b - F_{\hat{a} c} C_{\hat{c} \hat{b}} = i \delta_{\hat{a}}^\delta - \frac{2i}{\text{Im } F_{\hat{a} d} \bar{Z}^c Z^d} \text{Im } F_{\hat{a} e} \bar{Z}^c Z^d \]

(C.26)

and separating real and imaginary part

\[ A_{\hat{a}}^b = \text{Re } F_{\hat{a} c} C_{\hat{c} \hat{b}} + \delta_{\hat{a}}^\delta + i \text{ Im } F_{\hat{a} c} \left( C_{\hat{c} \hat{b}} - \frac{2}{\text{Im } F_{\hat{a} d} \bar{Z}^c Z^d} \bar{Z}^c Z^b \right). \] (C.27)

Likewise, one finds

\[ B_{\hat{a} \hat{b}} + F_{\hat{a} c} A_{\hat{c}}^b = -i F_{\hat{a} b} + \frac{2i}{\text{Im } F_{\hat{a} d} \bar{Z}^c Z^d} \text{Im } F_{\hat{a} e} \bar{Z}^c F_{\hat{b}} \]

(C.28)
Introducing a matrix

\[
\mathcal{M}_{\hat{a}\hat{b}} := \bar{\mathcal{F}}_{\hat{a}\hat{b}} + \frac{2i}{Z^a \text{Im} \mathcal{F}_{\hat{a}\hat{b}} Z^b} \text{Im} \mathcal{F}_{\hat{a}\hat{c}} \text{Im} \mathcal{F}_{\hat{b}\hat{d}} Z^d,
\]
(C.29)

we can write

\[
A = (\text{Re} \, \mathcal{M})(\text{Im} \, \mathcal{M})^{-1},
\]
\[
B = -(\text{Im} \, \mathcal{M}) - (\text{Re} \, \mathcal{M})(\text{Im} \, \mathcal{M})^{-1}(\text{Re} \, \mathcal{M}),
\]
\[
C = (\text{Im} \, \mathcal{M})^{-1}.
\]
(C.30)

We derive another identity which we use in the thesis:

\[
\mathcal{M}_{\hat{a}\hat{b}} Z^\hat{c} = \bar{\mathcal{F}}_{\hat{a}\hat{b}} Z^\hat{b} + 2i \text{Im} \mathcal{F}_{\hat{a}\hat{c}} Z^\hat{c}
\]
\[
= \mathcal{F}_{\hat{a}\hat{b}} Z^\hat{b},
\]
(C.31)

i.e.

\[
\mathcal{F}_{\hat{a}} = \mathcal{M}_{\hat{a}\hat{b}} Z^\hat{b}.
\]
(C.32)

For a prepotential of the form

\[
\mathcal{F} := -\frac{1}{3!} \mathcal{K}_{abc} \frac{Z^a Z^b Z^c}{Z^0}, \quad \mathcal{F}_{\hat{a}} := \partial_{Z^\hat{a}} \mathcal{F}, \quad Z = (1, t^A = b^A + iv^A),
\]
(C.33)

like the one for the Kähler moduli space in section 3.6, the matrix \( \mathcal{M} \) can be brought to an explicit form which we will now to. We have

\[
\mathcal{F}_{00} = -\frac{1}{3!} \partial_0 \mathcal{K}_{abc} \frac{Z^a Z^b Z^c}{-(Z^0)^2}
\]
\[
= -\frac{1}{3} \mathcal{K}_{abc} Z^a Z^b Z^c,
\]
\[
\mathcal{F}_{a0} = \frac{1}{3!} \mathcal{K}_{bcd} \partial_a \frac{Z^b Z^c Z^d}{(Z^0)^2}
\]
\[
= \frac{1}{2} \mathcal{K}_{abc} Z^b Z^c,
\]
\[
\mathcal{F}_{a\beta} = -\mathcal{K}_{abc} Z^c
\]
(C.34)
\( C.3 \) Gauge-coupling matrix

\[
\text{Im} \, \mathcal{F}_{ab} = - K_{abc} v^c = - K_{ab},
\]
\[
\text{Im} \, \mathcal{F}_{00} = - \frac{1}{3} \text{Im} \left[ (b^a + iv^a)(b^b + iv^b)(b^c + iv^c) \right]
\]
\[
= - \frac{1}{3} K_{abc} (v^a b^b c + v^b b^a c + v^c b^a b - v^a v^b v^c)
\]
\[
= - K_{ab} b^a b + \frac{1}{3} \mathcal{K} \quad \text{with} \quad \mathcal{K} = K_{a} v^a = K_{ab} v^a b = K_{abc} v^a v^b v^c,
\]
\[
\text{Im} \, \mathcal{F}_{ab} = \frac{1}{2} K_{abc} \text{Im} \left[ (b^a + iv^a)(b^b + iv^b) \right]
\]
\[
= \frac{1}{2} K_{abc} (b^c v^a + v^b v^c)
\]
\[
= K_{ab} b^a. \tag{C.35}
\]
Thus,
\[
Z^a \text{Im} \, \mathcal{F}_{ab} Z^b = \left( -K_{ab} b^a b + \frac{1}{3} \mathcal{K} \right) + 2 (K_{ab} b^a) Z^b + (-K_{ab} Z^a Z^b). \tag{C.36}
\]

With
\[
K_{ab} Z^a Z^b = K_{ab} (b^a + iv^a)(b^b + iv^b)
\]
\[
= K_{ab} b^a b + 2i K_{a} b^a - \mathcal{K},
\]
\[
K_{ab} b^a Z^b = K_{ab} b^a b + i K_{b} a,
\]
we have
\[
Z^a \text{Im} \, \mathcal{F}_{ab} Z^b = K_{ab} b^a b + \frac{1}{3} \mathcal{K} + 2i K_{a} b^a - K_{ab} b^a b - 2i K_{a} b^a + \mathcal{K}
\]
\[
= \frac{4}{3} \mathcal{K}. \tag{C.37}
\]

Thus,
\[
\mathcal{M}_{00} = - \frac{1}{3} K_{abc} (b^a - iv^a)(b^b - iv^b)(b^c - iv^c) + \frac{2i}{3 \mathcal{K}} (\text{Im} \, \mathcal{F}_{0b}) Z^b \tag{C.38}
\]

Thus,
\[
\mathcal{M}_{00} = - \frac{1}{3} K_{abc} (b^a - iv^a)(b^b - iv^b)(b^c - iv^c) + \frac{2i}{3 \mathcal{K}} (\text{Im} \, \mathcal{F}_{0b}) Z^b \tag{C.39}
\]
where

\[
i(\text{Im } F_{ab} Z^b)^2 = i(-\mathcal{K}_{ab} b^ab^b + \frac{1}{3} \mathcal{K} + \mathcal{K}_{ab} b^a Z^b)^2
\]

\[
= i \left( -\mathcal{K}_{ab} b^ab^b + \mathcal{K}_{ab} b^a b^b + \frac{1}{3} \mathcal{K} + i\mathcal{K}_{ab} b^a v^b \right)^2
\]

\[
= i \left( \frac{1}{3} \mathcal{K} + i\mathcal{K}_a \right)^2
\]

\[
= \left( \frac{1}{3} \mathcal{K} + i\mathcal{K}_a b^a \right) \left( i \frac{1}{3} \mathcal{K} - \mathcal{K}_a b^a \right).
\] (C.40)

The imaginary part is

\[
\text{Im } M_{00} = -\frac{1}{3} \mathcal{K}_{abc} (-3v^ab^c b^b + v^a v^b v^c) + \frac{3}{2\mathcal{K}} \left[ \left( \frac{1}{3} \mathcal{K} \right)^2 - \mathcal{K}_a b^a \mathcal{K}_{ab} b^b \right]
\]

\[
= \mathcal{K}_{ab} b^a b^b - \frac{1}{3} \mathcal{K} + \frac{\mathcal{K}}{6} - \frac{3}{2\mathcal{K}} \mathcal{K}_a b^a \mathcal{K}_{ab} b^b
\]

\[
= -\frac{\mathcal{K}}{6} \left( 1 - \frac{6}{\mathcal{K}} \mathcal{K}_{ab} + \frac{9}{\mathcal{K}^2} \mathcal{K}_a \mathcal{K}_b \right) b^a b^b
\]

\[
= -\frac{\mathcal{K}}{6} \left( 1 + 4G_{ab} b^a b^b \right).
\] (C.41)

The 0a-term is

\[
M_{0a} = \bar{F}_{0a} + \frac{3i}{2\mathcal{K}} (\text{Im } F_{ab} Z^b)(\text{Im } F_{0c} Z^c)
\] (C.42)

with

\[
(\text{Im } F_{ab} Z^b) = \text{Im } F_{a0} + \text{Im } F_{ab} Z^b
\]

\[
= \mathcal{K}_{ab} b^b - \mathcal{K}_a (b^b + iv^b)
\]

\[
= -i\mathcal{K}_a.
\] (C.43)
C.3 Gauge-coupling matrix

i.e.

\[ \text{Im} \mathcal{M}_{0a} = -\mathcal{K}_{ab} b^b + \text{Im} \left[ \frac{3i}{2\mathcal{K}} (-i\mathcal{K}_a) \left( \frac{1}{3} \mathcal{K} + i\mathcal{K}_b b^b \right) \right] \]

\[ = -\mathcal{K}_{ab} b^b + \frac{3}{2\mathcal{K}} \mathcal{K}_a \mathcal{K}_b b^b \]

\[ = -\frac{\mathcal{K}}{6} \left( -4\frac{3}{2\mathcal{K}} \right) \left( \mathcal{K}_{ab} - \frac{3}{2\mathcal{K}} \mathcal{K}_a \mathcal{K}_b \right) b^b \]

\[ = -\frac{\mathcal{K}}{6} (-4G_{ab} b^b). \quad (C.44) \]

The \( ab \)-term is

\[ \mathcal{M}_{ab} = \bar{\mathcal{F}}_{ab} + \frac{3i}{2\mathcal{K}} \text{Im} \mathcal{F}_{a\bar{c}} Z^\bar{c} \text{Im} \mathcal{F}_{b\bar{d}} Z^\bar{d} \]

\[ = -\mathcal{K}_{abc} (b^c - iv^c) + \frac{3i}{2\mathcal{K}} (-i\mathcal{K}_a) (-i\mathcal{K}_b) \quad (C.45) \]

and has imaginary part

\[ \text{Im} \mathcal{M}_{ab} = \mathcal{K}_{ab} - \frac{3}{2\mathcal{K}} \mathcal{K}_a \mathcal{K}_b \]

\[ = -\frac{\mathcal{K}}{6} \left( -4\frac{3}{2\mathcal{K}} \right) \left( \mathcal{K}_{ab} - \frac{3}{2\mathcal{K}} \mathcal{K}_a \mathcal{K}_b \right) \]

\[ = -\frac{\mathcal{K}}{6} 4G_{ab}. \quad (C.46) \]

Gathering the terms (C.49), (C.42) and (C.46), this can be written in matrix notation as

\[ \text{Im} \mathcal{M}_{\hat{a}\hat{b}} = -\frac{\mathcal{K}}{6} \begin{pmatrix} 1 + 4G_{ab} b^b b^b & -4G_{ab} b^b \\ -4G_{ab} b^b & 4G_{ab} \end{pmatrix}. \quad (C.47) \]

Now, we’ll compute

\[ \text{Re} \mathcal{M}_{\hat{a}\hat{b}} = \text{Re} \left( \bar{\mathcal{F}}_{\hat{a}\hat{b}} + \frac{2i}{Z^\hat{a} \text{Im} \mathcal{F}_{\hat{a}\hat{b}} Z^\hat{b}} \text{Im} \mathcal{F}_{\hat{a}\hat{c}} Z^\hat{c} \text{Im} \mathcal{F}_{\hat{b}\hat{d}} Z^\hat{d} \right). \quad (C.48) \]

To do so, recall that we already found

\[ \mathcal{M}_{00} = -\frac{1}{3} \mathcal{K}_{abc} (b^a - i\nu^a)(b^b - i\nu^b)(b^c - i\nu^c) + \frac{2i}{3\mathcal{K}} (\text{Im} \mathcal{F}_{0\hat{b}} Z^\hat{b})^2 \quad (C.49) \]
where
\[ i(\text{Im } F_{0b}Z^b)^2 = \left( \frac{1}{3} \mathcal{K} + i\mathcal{K}_a b^a \right) \left( \frac{1}{3} \mathcal{K} - \mathcal{K}_a b^a \right). \] (C.50)

Thus,
\[
\text{Re } M_{00} = -\frac{1}{3} \mathcal{K}_{abc} b^a b^b c + \mathcal{K}_a b^a \frac{3}{2\mathcal{K}} \left( -\frac{1}{3} \mathcal{K} \mathcal{K}_a b^a - \frac{1}{3} \mathcal{K} \mathcal{K}_a b^a \mathcal{K} \right) \\
= -\frac{1}{3} \mathcal{K}_{abc} b^a b^b c + \mathcal{K}_a b^a - \mathcal{K}_a b^a \\
= -\frac{1}{3} \mathcal{K}_{abc} b^a b^b c. \] (C.51)

In order to compute the real part of
\[ M_{0a} = \bar{F}_{0a} + \frac{3i}{2\mathcal{K}} (\text{Im } F_{ab} Z^b)(\text{Im } F_{0c} Z^c), \] (C.52)
we note that
\[
\text{Re } F_{0a} = \frac{1}{2} \text{Re } \left[ \mathcal{K}_{abc} (b^b + iv^b)(b^c + iv^c) \right] \\
= \frac{1}{2} \mathcal{K}_{abc} (b^b b^c - v^b v^c) \\
= \frac{1}{2} \mathcal{K}_{abc} b^b b^c - \frac{1}{2} \mathcal{K}_a. \] (C.53)
and recall
\[ (\text{Im } F_{ab} Z^b) = -i\mathcal{K}_a. \] (C.54)
Thus,
\[
\text{Re } M_{0a} = \frac{1}{2} \mathcal{K}_{abc} b^b b^c - \frac{1}{2} \mathcal{K}_a + \frac{3}{2\mathcal{K}} \text{Re } \left[ i(-i\mathcal{K}_a) \left( \frac{1}{3} \mathcal{K} + i\mathcal{K}_b b^b \right) \right] \\
= \frac{1}{2} \mathcal{K}_{abc} b^b b^c - \frac{1}{2} \mathcal{K}_a + \frac{3}{2\mathcal{K}} \mathcal{K}_a \frac{1}{3} \mathcal{K} \\
= \frac{1}{2} \mathcal{K}_{abc} b^b b^c. \] (C.55)
The \( ab \)-term is

\[
\mathcal{M}_{ab} = \mathcal{F}_{ab} + \frac{3i}{2\mathcal{K}} \text{Im} \mathcal{F}_{ac} Z^c \text{Im} \mathcal{F}_{bd} Z^d
\]

\[
= -\mathcal{K}_{abc}(b^c - iv^c) + \frac{3i}{2\mathcal{K}} (-i\mathcal{K}_a)(-i\mathcal{K}_b)
\]

(C.56)

and has real part

\[
\text{Re} \mathcal{M}_{ab} = -\mathcal{K}_{abc} b^c.
\]

(C.57)

Gathering all terms, we have

\[
\text{Re} \mathcal{M}_{\hat{a}\hat{b}} = \left( -\frac{1}{3} \mathcal{K}_{cde} b^c b^d b^e + \frac{1}{2} \mathcal{K}_{acd} b^c b^d b^e \right)
\]

(C.58)

After this lengthy calculations, we finally arrive at the explicit expression

\[
\mathcal{M} = \left( \begin{array}{ccc}
-\frac{1}{3} \mathcal{K}_{cde} b^c b^d b^e & \frac{1}{2} \mathcal{K}_{acd} b^c b^d b^e & -\frac{1}{6} \mathcal{K} \\
\frac{1}{2} \mathcal{K}_{acd} b^c b^d & -\mathcal{K}_{abc} b^c & -4G_{ab} b^b \\
1 + 4G_{ab} b^b & -4G_{ab} b^b & 4G_{ab}
\end{array} \right).
\]

(C.59)
REFERENCES


Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den 4. August, 2018