Introduction to Renormalization Group

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QUANTUM FIELD THEORY

QUANTUM MECHANICS:

The wave function $\psi_t(x)$ - probability amplitude to find a particle at point x at time t.

SHROEDINGER EQUATION:

$$i\frac{d}{dt}\psi = H\psi$$

QUANTUM FIELD THEORY:

A FINITE NUMBER OF QM DEGREES OF FREEDOM AT EVERY SPATIAL POINT $\phi(x)$

WAVE FUNCTIONAL $\Psi_t[\phi(x)]$ - PROBABILITY AMPLITUDE DENSITY TO FIND A GIVEN FIELD CONFIGURATION $\phi(x)$ at time t.

THERE IS STILL SCHROEDINGER EQUATION, BUT IT IS MUCH LESS USEFUL

$\Psi[\phi]$ contains too much information, even knowing Ψ it is still hard work to get this information out.
USUALLY WE ARE CONTENT WITH LESS.

AS PARTICLE PHYSICISTS WE WANT SCATTERING AMPLITUDES.

AS CONDENSED MATTER PEOPLE WE WANT CORRELATION FUNCTIONS.

GREEN'S FUNCTIONS

$$D(x_1..x_n) = \langle 0|T\phi(x_1)...\phi(x_n)|0\rangle$$

Here $|0\rangle$ is the vacuum state of the theory; T is the time ordered product, which I will suppress in the following.

GREEN'S FUNCTIONS TELL US MOST OF WHAT WE EVER WANT TO KNOW:

SCATTERING MATRIX IS CALCULABLE FROM $D's$;

MUCH INFORMATION ABOUT FIELD DISTRIBUTIONS IN THE VACUUM;

REACTION OF THE VACUUM TO EXTERNAL PERTURBATIONS...

WE ARE ALWAYS DEALING WITH RELATIVISTIC FIELD THEORIES

THIS STILL LEAVES MANY OPTIONS FOR FUNDAMENTAL FIELDS. THEY CAN BE SCALAR (SPIN 0) $\phi(x)$, FERMIONS (SPIN 1/2) $\psi(x)$; OR VECTOR (SPIN 1) $A_\mu(x)$.

CALCULATING GREEN'S FUNCTIONS

A THEORY IS DEFINED BY ITS ACTION (TAKE A SIMPLE THEORY OF ONE SCALAR FIELD)

$$S = \int_{x,t} \left[\frac{1}{2}\partial_\mu \phi(x,t)\partial^\mu \phi(x,t) - \frac{1}{2}m^2\phi(x,t)^2 - \frac{\lambda}{4!}\phi(x,t)^4\right]$$

FUNCTIONAL INTEGRAL APPROACH

$$D^n(\phi(x_1)...\phi(x_n)) = \frac{\int D[\phi]\phi(x_1)...\phi(x_n)e^{iS}}{\int D[\phi]e^{iS}}$$

INFINITELY DIMENSIONAL INTEGRAL - EASIER SAID THAN DONE.

FOR FREE FIELDS $\lambda = 0$ THE INTEGRAL IS GAUSSIAN AND EXACTLY SOLVABLE.

THERE IS ONLY ONE NONTRIVIAL CONNECTED GREEN'S FUNCTION:

$$D^2(x_1 - x_2) = \int \frac{d^4p}{(2\pi)^4} e^{-ip_\mu x^\mu} \frac{i}{p^2 - m^2 + i\epsilon}$$

$\epsilon \to 0$ is a regulator, and although it is very important, I will disregard it most of the time.

WHEN WE TRANSLATE THIS TO THE LANGUAGE OF PARTICLES, THIS IS THE THEORY OF ONE SPECIES OF SCALAR PARTICLE WITH MASS m.

THE EIGENSTATES OF THE HAMILTONIAN (IF WE WERE TO WRITE THEM EXPLICITLY) ARE STATES WITH n PARTICLES OF ARBITRARY MOMENTUM $\vec{p}$ WITH DISPERSION RELATION $E_p = (p^2 + m^2)^{1/2}$

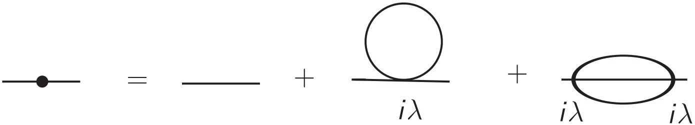
INTERACTING THEORY: FOR WEAK COUPLING ($\lambda \ll 1$) PERTURBATIVE CALCULATIONS USING FEYNMAN DIAGRAMS.

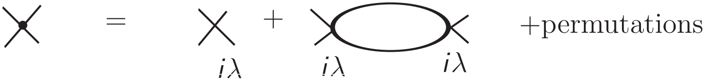
FEYNMAN RULES IN MOMENTUM SPACE

- For each propagator $D(p) = \frac{i}{p^2 - m^2 + i\epsilon}$ a line.
- For each vertex $i\lambda$ a cross.
- In order λ^n draw all (topologically distinct) diagramms that can be constructed from n vertices connected by propagators with given number of external points.
- Energy-Momentum conservation in each vertex
- Integrate over all loop momenta.
- Symmetry factors.

THE RULES FOLLOW DIRECTLY FROM EXPANSION OF THE FUNCTIONAL INTEGRAL IN POWERS OF λ.

HOWEVER IF WE TRY TO CALCULATE EVEN THE SIMPLEST OF THESE GRAPHS WE FIND A PROBLEM.





THE SIMPLEST CORRECTION TO THE PROPAGATOR INVOLVES THE INTEGRAL

$$\lambda \int_0^{\Lambda} \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2} \sim \lambda \Lambda^2 \to_{\Lambda\to\infty} \infty$$

THE CORRECTION TO THE FOUR POINT FUNCTION INVOLVES

$$\lambda^2 \int_0^{\Lambda} \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - m^2)((k+p)^2 - m^2)} \sim \lambda^2 \ln \frac{\Lambda^2}{k^2} \to_{\Lambda\to\infty} \infty$$

THE PROBLEM IS SEEMINGLY WELL POSED, THE RULES OF CALCULATION ARE WELL DEFINED, BUT OUT COME THE DIVERGENCIES.

WHAT IS GOING ON?

RENORMALIZATION.

FUNDAMENTALLY THE ANSWER IS:

THE PARAMETERS WE WRITE IN THE ACTION (m AND λ) ARE NOT DIRECTLY PHYSICALLY MEASURABLE QUANTITIES.

WE MEASURE MASSES OF PHYSICAL PARTICLES AND THEIR SCATTERING AMPLITUDES, RATHER THAN THE "BARE PARAMETERS".

TO MAKE SENSE OF THE THEORY, WE HAVE TO CALCULATE PHYSICAL QUANTITIES AND REQUIRE THAT THEY ARE FINITE.

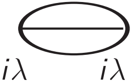
THE "BARE PARAMETERS" THEN CAN DEPEND ON THE UV CUTOFF Λ, AS LONG AS PHYSICAL OBSERVABLES DO NOT.

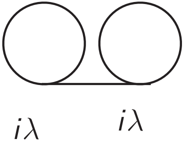
FOR EXAMPLE WE SHOULD CALCULATE $D^2(p)$ AND $D^4(p_i)$ AND EQUATE THEM TO THE QUANTITIES WE MEASURE IN THE EXPERIMENT (PARTICLE MASS AND SCATTERING AMPLITUDE), WHICH ARE OF COURSE FINITE NUMBERS. THESE ARE CALLED NORMALIZATION CONDITIONS.

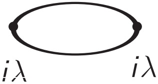
THE BARE COUPLINGS m_0^2 AND λ_0 ARE THEN MADE CUTOFF DEPENDENT, IN SUCH A WAY THAT THE NORMALIZATION CONDITIONS ARE SATISFIED.

FOR OUR THEORY THIS SCHEMATICALLY MEANS:

$m^2_{physical} = \quad m_0^2 \; +$ 

$+$ 

$+$ 

$i\lambda_{physical} \quad = \quad i\lambda_0 \quad +$ 

$+$permutations

$$m^2_{physical} = m_0^2 + \lambda_0 \Lambda^2 + \ldots; \quad \lambda_{physical} = \lambda_0 + \lambda_0^2 \ln \frac{\Lambda^2}{m^2} + \ldots$$

THESE OBSERVABLES MUST REMAIN FINITE IN THE LIMIT $\Lambda^2/m^2_{physical} \to \infty$.
THIS SHOULD HOLD FOR ANY MOMENTUM DEPENDENT GREENS FUNCTION WHEN MOMENTUM IS MUCH SMALLER THAN THE CUTOFF $\Lambda^2/p^2 \to \infty$.
TO MAKE THESE FINITE, WE NEED

$$\lambda_0(\Lambda) \sim \frac{1}{\ln \frac{\Lambda^2}{m^2}}; \qquad m_0^2 \sim -\lambda \Lambda^2$$

- A THEORY IS RENORMALIZABLE IF $D^n(p_i)$ AT MOMENTA $p^2 \ll \Lambda^2$ DO NOT DEPEND ON Λ.
- THEN THE CUTOFF CAN BE REMOVED BY SPECIFYING A FINITE NUMBER OF NORMALIZATION CONDITIONS, AND THE THEORY HAS PREDICTIVE POWER.

WHEN DOES THIS HAPPEN?

PERTURBATIVE ANSWER: WHEN THE DIMENSION OF THE COUPLING CONSTANT IN UNITS OF MASS IS NONNEGATIVE: $[\lambda] \equiv \omega \geq 0$.
Roughly: perturbative expansion is in powers of a dimensionless quantity $\lambda\Lambda^{-\omega}$. If $\omega < 0$, there are more divergencies order by order in PT and one needs infinite number of NORMALIZATION CONDITIONS to banish them all.

HAVE WE SAVED PT?

SO SUPPOSE OUR THEORY SATISFIES CONDITIONS FOR RENORMALIZABILITY.

CAN WE NOW DO PT? NOT REALLY!

NAIVELY THE BARE PARAMETERS HAVE CUTOFF DEPENDENCE:

$$m_0^2 \sim \Lambda^2; \qquad \lambda_0 \sim \frac{1}{\ln \frac{\Lambda^2}{m^2}} \qquad (\mathrm{actually}\, \lambda_0 \sim \ln \frac{\Lambda^2}{m^2})$$

IN PT WE HAVE:

$$\lambda(\mu) = \lambda_0 \left[1 + x\lambda_0 \ln \frac{\Lambda^2}{m^2} + y\lambda_0^2 \ln^2 \frac{\Lambda^2}{m^2} + ... \right]$$

SO WE CANNOT CALCULATE $\lambda(\mu)$ IN PT IN λ_0. EVEN IF THEY ARE BOTH SMALL, THEY ARE NOT RELATED PERTURBATIVELY.

WHY IS THE CORRECTION SO BIG?

$$\delta\lambda(k^2) = -\lambda^2 \int^{\Lambda} \frac{d^4p}{(2\pi)^4} \frac{1}{p^2(k-p)^2} \sim \lambda^2 \ln \frac{\Lambda^2}{k^2}$$

THE LARGE LOGARITHM COMES FROM THE LOOP MOMENTA WHICH ARE MUCH LARGER THAN THE PHYSICAL MOMENTA AT WHICH WE WANT TO MEASURE λ; $|p| \gg |k|$.

THE CORRECTION IS LARGE NOT BECAUSE THE INTERACTION IS STRONG, BUT BECAUSE THERE IS A LOT OF PHASE SPACE TO INTEGRATE OVER. IF WE ONY INTEGRATED UP TO, SAY $\mu = 10^3|k|$, WE WOULD GET

$$\delta\lambda(k^2) \sim -\lambda^2 \ln \frac{\mu^2}{k^2} = -6\lambda^2$$

A SIZABLE CORRECTION, BUT STILL PERTURBATIVELY SMALL IF $\lambda \ll 1$.

WILSON RENORMALIZATION GROUP

LET US SUCCESSIVELY "INTEGRATE OUT" THE UV DEGREES OF FREEDOM SLICE BY SLICE IN MOMENTUM SPACE.

EACH STEP IS PERTURBATIVE IN THE COUPLING CONSTANT

ONCE WE INTEGRATE OUT THE SLICE BETWEEN Λ AND Λ', THE REMAINING DEGREES OF FREEDOM WILL HAVE DIFFERENT INTERACTION STRENGTH $\lambda(\Lambda')$.

FOR THE NEXT STEP IN INTEGRATION WE WILL EXPAND IN $\lambda(\Lambda')$.

THIS "INTEGRATION OUT" ACCOMPANIED BY THE CHANGE OF COUPLING CONSTANTS IS CALLED RENORMALIZATION GROUP TRANSFORMATION

WE DO THIS UNTIL WE GET CLOSE TO THE SCALE μ AT WHICH WE WANT TO CALCULATE THE CORRELATORS, AND THEN USE PT IN $\lambda(\mu)$

THUS WE ALWAYS EXPAND IN $\lambda(\mu)$ AND NOT $\lambda_0(\Lambda)$ AND ARE LEGAL AS LONG AS $\lambda(\mu) \ll 1$

LET US SEE HOW IT WORKS.

$$Z = \int [D\phi] e^{i \int d^4x [\frac{1}{2}(\partial^\mu \phi)^2 - \frac{1}{2} m_0^2 \phi^2 - \frac{\lambda_0}{4!}\phi^4]}$$

SPLIT THE FIELD INTO HIGH AND LOW MOMENTUM MODES

$\phi(k) = \phi_L(k) + \phi_H(k);\quad \phi_L(k) = 0;\ k < \Lambda_1;\quad \phi_H(k) = 0;\ k > \Lambda_1;\quad D[\phi] = \Pi_k d\phi_L(k) d\phi_H(k)$

NOW INTEGRATE PERTURBATIVELY OVER $\phi_H(k)$ KEEPING $\phi_L(k)$ FIXED

THIS PROCEDURE RENORMALIZES OLD VERTICES AND GENERATES NEW VERTICES IN THE REMAINING PATH INTEGRAL OVER ϕ_L

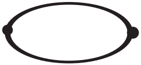
THE INTERACTION TERM CONTAINS INTERACTION BETWEEN ϕ_L AND ϕ_H

$$\lambda\phi^4 \to \lambda\phi_L^4 + 6\lambda(\phi_L)^2(\phi_H)^2 + ...$$

$$\Lambda_1 < p < \Lambda$$

$-\frac{\lambda}{4}\phi_L^2 < \phi_H^2 >= -\frac{\lambda}{4}\phi_L^2$  ; $\quad \delta m_L^2 = \frac{\lambda}{32\pi^2}(\Lambda^2 - \Lambda_1^2)$

ANALOGOUSLY THE INTERACTION GETS RENORMALIZED AS

$$\delta\lambda_L =$$  $$= -\frac{3\lambda^2}{16\pi^2}\ln\frac{\Lambda}{\Lambda_1}$$

THIS PROCEDURE GENERATES OTHER TERMS WHICH WHERE NOT IN THE ORIGINAL ACTION, FOR EXAMPLE THE SIX POINT VERTEX AT ONE LOOP PROPORTIONAL TO λ^3.

$$\delta S = \int d^4x\xi\phi_L^6; \qquad \xi \propto \lambda^3\int_{\Lambda_1}^{\Lambda}\frac{d^4p}{(2\pi)^4}\frac{1}{p^6} \sim \frac{\lambda^3}{\Lambda^2}$$

SO FOR ϕ_L WE HAVE A MODIFIED "EFFECTIVE ACTION" WHICH NOW SERVES AS THE WEIGHT IN THE FUNCTIONAL INTEGRAL

$$S = \int d^4x\Big[\frac{1}{2}Z(\Lambda_1)(\partial^\mu\phi_L)^2 - \frac{1}{2}m(\Lambda_1)^2\phi_L^2 - \frac{\lambda(\Lambda_1)}{4!}\phi_L^4 + \text{other interactions}\Big]$$

LET US TRY TO UNDERSTAND THE GENERAL PROPERTIES OF THE CHANGE OF THE ACTION UNDER RG TRANSFORMATION.

FIRST OF ALL IT IS CONVENIENT TO BRING THE ACTION TO A STANDARD FORM.
WE RESCALE THE COORDINATES, SO THAT THE UV CUTOFF IN THE NEW ACTION IS THE SAME AS IN THE OLD ONE

$$p \to p' = \frac{\Lambda}{\Lambda_1} p; \qquad x \to x' = \frac{\Lambda_1}{\Lambda} x$$

WE ALSO RESCALE THE FIELD ITSELF SO THAT IT'S KINETIC TERM HAS A CANONICAL COEFFICIENT 1/2, $\phi_L \to Z^{-1}\phi_L$.

$$S_L = \int d^4x \left[\frac{1}{2}(\partial_\mu \phi_L)^2 - \frac{1}{2} m_1^2 \phi_L^2 - \frac{\lambda_1}{4!}\phi_L^4 - \xi_1 \phi_L^6 + \ldots\right]$$

WITH

$$m_1^2 = m(\Lambda_1)^2 Z^{-1}\frac{\Lambda^2}{\Lambda_1^2}; \quad \lambda_1 = \lambda(\Lambda_1) Z^{-2}; \quad \xi_1 = \xi(\Lambda_1) Z^{-3}\frac{\Lambda_1^2}{\Lambda^2}$$

AS WE EXAMINE THE THEORY AT A LOWER SCALE, THE COUPLINGS "FLOW" WITH THE SCALE. PERTURBATIVELY THE DIRECTION OF THE FLOW IS DETERMINED BY THE DIMENSION OF THE COUPLING CONSTANT.

DEFINE THE RG PARAMETER $t = \ln \frac{\Lambda_1}{\Lambda}$

THEN

$$\frac{\partial m^2}{\partial t} = -2m^2 + O(\lambda); \quad \frac{\partial \lambda}{\partial t} = O(\lambda^2); \quad \frac{\partial \xi}{\partial t} = 2\xi + O(\lambda^3)$$

DECREASING t IS FLOWING INTO THE INFRARED.

MASS INCREASES TOWARD IR, WHILE ξ DECREASES.

- COUPLINGS THAT INCREASE IN IR- **RELEVANT**;
- COUPLINGS THAT DECREASE IN IR - **IRRELEVANT**;
- COUPLINGS THAT DO NOT CHANGE - **MARGINAL**; (MARGINALLY RELEVANT OR MARGINALLY IRRELEVANT)

RELEVANT COUPLINGS DETERMINE THE NATURE OF THE THEORY IN THE INFRARED.

m^2 - RELEVANT; ξ - IRRELEVANT; λ - MARGINAL;

TO FIND HOW λ BEHAVES WE NEED TO ACTUALY CALCULATE THE ONE LOOP CORRECTION. IT TURNS OUT TO BE

$$\lambda_1 = \lambda - \frac{3\lambda^2}{16\pi^2} \ln \frac{\Lambda}{\Lambda_1}$$

OR IN TERMS OF EVOLUTION

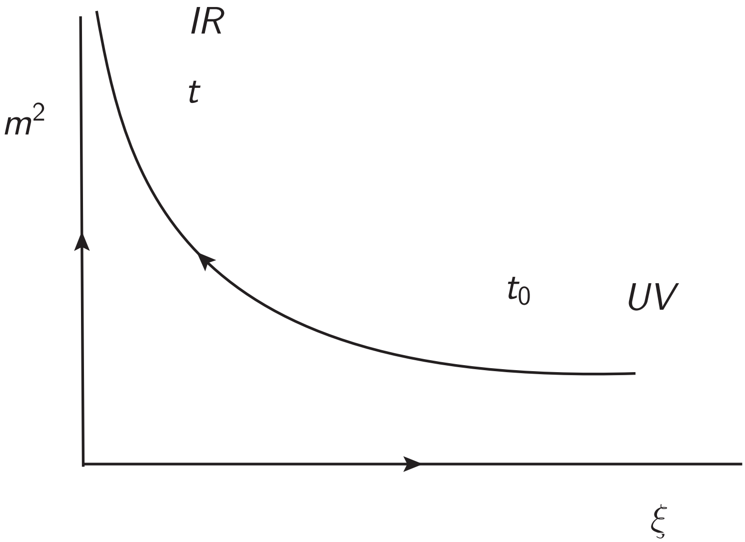
$$\frac{\partial \lambda}{\partial t} \equiv \beta(\lambda) = \frac{3}{16\pi^2}\lambda^2 > 0$$

SO λ IS MARGINALLY IRRELEVANT COUPLING

THE SOLUTION OF RG EQUATIONS:

$$m^2(t) = m^2(t_0)e^{-2(t-t_0)}; \qquad \xi(t) = \xi(t_0)e^{2(t-t_0)};$$

$$\lambda(t) = \frac{\lambda(t_0)}{1 - \frac{3}{16\pi^2}\lambda(t_0)(t-t_0)}$$


IR
t
m^2
t_0
UV
ξ

INFRARED FIXED POINT

Q: WHERE DOES THE THEORY FLOW IN THE ULTIMATE IR?
THE SOLUTION TELLS US

$$m^2 \to \infty; \quad \xi \to 0; \quad \lambda \to 0$$

THIS IS THE FIXED POINT OF THE FLOW - THE THEORY DOES NOT GET OUT OF THERE.

THE INFINITE MASS LIMIT IS EASY TO UNDERSTAND. IF THE MOMENTUM IS MUCH SMALLER THAN THE MASS, IT LOOKS LIKE THE MASS IS INFINITE. FOR ANY, EVEN SMALL MASS, AT ZERO MOMENTUM THE THEORY IS EQUIVALENT TO INFINITELY MASSIVE THEORY.

ZERO ξ: ANY IRRELEVANT COUPLING DIES AND CAN BE FORGOTTEN IN THE IR.

FINALLY λ ALSO DIES, SINCE IT TURNED OUT TO BE MARGINALLY IRRELEVANT, BUT IT DIES MUCH SLOWER THAN ξ. SCALAR THEORY IN 4 DIMENSIONS IS STRICTLY SPEAKING NONINTERACTING IN THE IR. IN OTHER WORDS, IF WE START WITH ANY FINITE COUPLING AT THE UV CUTOFF SCALE Λ, IN THE LIMIT $\Lambda \to \infty$, THE PHYSICAL COUPLING VANISHES.

THIS IS THE TRIVIAL NONINTERACTING, INFINITELY MASSLESS FIXED POINT OF THE RG FLOW.

TRIVIALITY OF $\lambda\phi^4$

TO SEE THIS, LET US EXAMINE THE SOLUTION:

$$\lambda(t) = \frac{\lambda(t_0)}{1 - \frac{3}{16\pi^2}\lambda(t_0)(t - t_0)}$$

IF $\lambda(t_0)$ IS FINITE AT $t_0 \to \infty$, THEN, AT ANY FINITE t; $\lambda(t) = 0$.
IF t_0 IS FINITE, THAN THE COUPLING DIVERGES AT

$$t - t_0 = \frac{16\pi^2}{3\lambda(t_0)}; \qquad \mu_{Landau} = \mu_0 e^{\frac{16\pi^2}{3\lambda(\mu_0)}}$$

THIS IS THE SO CALLED LANDAU GHOST (OR LANDAU POLE).

SIMILAR PROBLEM AFFLICTS QED - QUANTUM ELECTRODYNAMICS.

IN THE STRICT SENSE THE SCALAR THEORY (AND QED) IS TRIVIAL - IT IS NONINTERACTING IN THE IR.

HOWEVER THE LANDAU GHOST SCALE IS HUMONGOUS.

IN QED, $\mu_{Landau} \approx 10^{280} Gev$, WHILE WE KNOW THAT ALREADY AT $10^{19} Gev$ GRAVITY BECOMES STRONG, AND WILL STRONGLY AFFECT OUR ANALYSIS.

THUS WE SHOULD NOT TAKE SERIOUSLY ANY PROBLEM THAT ARISES ABOVE THE PLANCK MASS SCALE.

DIFFERENT IR BEHAVIOR

IS INFINITELY MASSIVE BEHAVIOR IS THE ONLY POSSIBLE ONE IN THE INFRARED?

CERTAINLY NOT.

EASIEST SEEN FOR A THEORY WITH MORE THAN ONE SCALAR FIELD AND A CONTINUOUS GLOBAL SYMMETRY.

TAKE TWO SCALARS $\phi_i, \quad i = 1, 2$

$$S = \int \left[\frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{1}{2} m^2 \phi_i^2 - \frac{\lambda}{4!}(\phi_i^2)^2\right]$$

FOR $m^2 < 0$ THE $O(2)$ SYMMETRY IS SPONTANEOUSLY BROKEN BY THE EXPECTATION VALUE OF THE FIELD $< \phi_1 >= f_\pi$.

GOLDSTONE THEOREM - THERE IS A MASSLESS PARTICLE - GOLDSTONE BOSON (THE PHASE OF THE FIED ϕ).

INTERACTIONS OF GB ARE SOFT - COUPLING DECREASES AS p^2/f_π^2.

WITHOUT MUCH CALCULATION IT IS CLEAR THAT THE IR FIXED POINT OF THIS THEORY IS A MASSLESS NONINTERACTING SCALAR FIELD, NOT AN INFINITELY MASSIVE ONE.

ANALOGY WITH STATISTICAL MECHANICS.

WE HAVE INTRODUCED THE CONCEPT OF RG IN THE CONTEXT OF PERTURBATION THEORY, BUT IT IS CLEARLY NOT RESTRICTED TO WEAKLY COUPLED THEORIES.

IN FACT THE FIRST IDEA OF RG TRANSFORMATION IS DUE TO KADANOFF, AND IS IN THE CONTEXT OF STATISTICAL MECHANICAL SYSTEMS.

THERE IS AN INTIMATE RELATION BETWEEN THE QFT AND STATISTICAL MECHANICS.
ON THE FORMAL LEVEL, RECALL THAT WE HAVE TO CALCULATE LOOP INTEGRALS OF THE FORM

$$\int \frac{d^4 p}{p^2 - m^2} = \int \frac{d^4 p}{p_0^2 - \vec{p}^2 - m^2}$$

TO PERFORM THIS INTEGRAL WE CAN EITHER INTRODUCE THE $i\epsilon$ REGULATOR, OR PERFORM WICK ROTATION IN THE p_0 COMPEX PLANE, WHICH AMOUNTS TO SUBSTITUTION

$$p_0 \to i p_0$$

IN COORDINATE SPACE THIS IS EQUIVALENT TO

$$x_0 \to -ix_0$$

UNDER THIS TRANSFORMATION

$$Z \to \int [D\phi] e^{-S_E}; \qquad S_E = \int d^4x \Big[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!}(\phi^2)^2\Big]$$

THIS LOOKS LIKE A STATISTICAL SUM OF A CLASSICAL FIELD THEORY IN 4 SPATIAL DIMENSIONS WITH ENERGY FUNCTIONAL S_E.

THIS CORRESPONDENCE IS VERY GENERAL:

FUNCTIONAL INTEGRAL OF A RELATIVISTIC QFT IN $1+d$ DIMENSIONS IS EQUIVALENT TO A STATISTICAL SUM OF A CLASSICAL STATISTICAL SYSTEM IN $1+d$ SPATIAL DIMENSIONS.

OUR ϕ^4 ACTION IS IN THE SAME UNIVERSALITY CLASS AS THE ISING MODEL.

THE $O(2)$ INVARIANT ϕ^4 IS SIMILARLY RELATED TO THE EASY AXIS HEISENBERG FERROMAGNET.

FIXED POINTS IN STAT MECH

IN STAT MECH THE FIXED POINTS WE DISCUSSED HAVE A VERY NATURAL INTERPRETATION.
ANY STAT MECH SYSTEM WHICH HAS SECOND ORDER PHASE TRANSITION AT SOME T_C, HAS THREE DISTINCT REGIMES OF LONG DISTANCE (LOW MOMENTUM) BEHAVIOR

- HIGH T: $T \gg T_C$;
- LOW T: $T \ll T_C$;
- CRITICAL: $T = T_C$;

CONSIDER FERROMAGNET AT VERY HIGH TEMPERATURE.

HERE ENTROPY OVERWHELMS ENERGY.

SPINS FLUCTUATE RANDOMLY FROM POINT TO POINT, AND THERE IS ABSOLUTELY NO CORRELATION BETWEEN SPINS AT DIFFERENT SPATIAL POSITIONS.

$$< \phi(x)\phi(y) >= 0; \qquad x \neq y; \qquad D^2(p) = 0$$

THIS IS THE TRIVIAL, INFINITELY MASSIVE FIXED POINT $D^2(p) \sim \frac{1}{p^2+m^2} \to_{m\to\infty} 0$.

FAR BELOW T_C THE $O(2)$ SYMMETRY IS SPONTANEOUSLY BROKEN. SPINS ARE STRONGLY CORRELATED WITH INFINITE CORRELATION LENGTH

$$< \phi(x)\phi(y) > \to_{|x-y|\to\infty} const$$

.

ABOVE THIS CONSTANT BACKGROUND THE ONLY PROPAGATING EXCITATION AT LARGE DISTANCES IS THE MASSLESS MAGNON. SUCH A FIXED POINT CORRESPONDS TO $m^2 \to -\infty$.
FOR $m^2 < 0$

$$S_E = \int \left[\frac{1}{2}(\partial_\mu \phi_i)^2 + \frac{1}{2} m^2 \phi_i^2 + \frac{\lambda}{4!}(\phi_i^2)^2\right]$$

THE CLASSICAL VACUUM IS AT

$$< \phi^2 >= \frac{-6m^2}{\lambda}$$

PARAMETRIZING

$$\phi_1 = \rho \cos \chi; \qquad \phi_2 = \rho \sin \chi$$

WE FIND THAT χ IS FREE MASSLESS BOSON; WHILE ρ IS INTERACTING, BUT HAS THE MASS $M^2 = -2m^2 \to \infty$.
THE FREE MASSLESS FP IN STAT MECH LANGUAGE IS THE LOW TEMPERATURE FP

THE SIGN OF THE MASS TERM IS CORRELATED (AT LEAST PERTURBATIVELY) WITH THE SIGN OF $T - T_C$.

INDEED IN STAT MECH CONTEXT THE EUCLIDEAN ACTION USUALLY APPEARS WITH $m^2 = \Lambda(T - T_C)$

NOTE: BOTH FIXED POINTS CORRESPOND TO $|m^2| \to \infty$ CONSISTENT WITH THE FACT THAT m^2 IS A RELEVANT COUPING, AND THUS GROWS INTO IR.

THE $T = 0$ AND $T \to \infty$ FP's ARE INFRARED STABLE - SMALL CHANGE IN BARE PARAMETERS DOES NOT CHANGE IR BEHAVIOR.

WHAT HAPPENS WHEN THE THEORY IS EXACTLY CRITICAL?

THERE IS AN IR UNSTABLE FP: IF WE TUNE THE BARE PARAMETER m^2 TO EXACTLY THE RIGHT VALUE, IT WILL NOT CHANGE AS WE MOVE INTO IR.

CLASSICALLY (PERTURBATIVELY) THIS VALUE IS $m^2 = 0$,

BUT THINGS ARE A LITTLE MORE COMPLICATED (MORE - LATER).

CRITICALITY

WHAT IS THIS UNSTABLE FP?
IN $d = 4$ IT IS EASY TO UNDERSTAND.

- MASSLESS: $m^2 = 0$ IS AN UNSTABLE SOLUTION OF $m^2(t) = m^2(t_0)e^{-2(t-t_0)}$
- $\lambda = 0$ IS THE ONLY ZERO OF THE β-function.
- FOR FERROMAGNET IT IS THE THEORY OF TWO MASSLESS FIELDS ϕ_i - THE $O(2)$ SYMMETRY REMAINS UNBROKEN.

SIMILAR BUT NOT THE SAME AS ZERO TEMPERATURE FIXED POINT.

SIMILARITY IS AN ACCIDENT OF 4D. IN ANY $d < 4$ THE β-function HAS TWO ZEROES

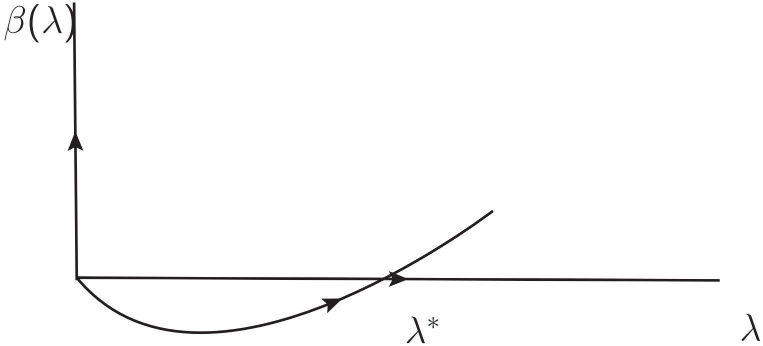
$$\frac{\partial\lambda}{\partial t} \equiv \beta(\lambda) = (d-4)\lambda + \frac{3}{16\pi^2}\lambda^2$$

THE FIRST TERM BECAUSE λ HAS DIMENSION $4-d$ IN d-DIMENSIONS

THIS HAS TWO FIXED POINTS

- $\lambda = 0$ - IR UNSTABLE, UV STABLE
- $\lambda^* = \frac{(4-d)16\pi^2}{3}$ - IR STABLE, UV UNSTABLE WILSON-FISCHER FIXED POINT

THE FLOW $d < 4$ DIMENSIONS



SCALE INVARIANCE OF FIXED POINT

A FIXED POINT IS BY DEFINITION SCALE INVARIANT. FIXED POINT OF RG FLOW HAS ALL β-FUNCTIONS VANISHING

$$\frac{\partial \lambda_i}{\partial t} \equiv \beta_i(\lambda_j) = 0; \quad for \quad all \quad i$$

AT A FP NONE OF THE COUPINGS CHANGE WITH SCALE - THE THEORY IS THE SAME AT ALL SCALES, AND SO SCALE INVARIANT

THIS DOES NOT MEAN THAT THE THEORY IS TRIVIAL - CORRELATION FUNCTIONS MAY SCALE WITH A DIFFERENT SCALING DIMENSION THAN IN THE FREE THEORY.

IN PARTICULAR ONE DEFINES SCALING EXPONENTS VIA

$$<\phi(p)\phi(-p)>= \frac{\mu^{-\eta}}{p^{2-\eta}}; \qquad <\phi^2(p)\phi^2(-p)>= \mu^{d-4}\left(\frac{\mu}{p}\right)^{d-2-\frac{1}{\nu}}$$

CRITICAL EXPONENTS OF SECOND ORDER PHASE TRANSITIONS ARE CALCULABLE IN TERMS OF THESE "CRITICAL INDICES" $\eta, \quad \nu$ THEY WERE CALCULATED FOR THE WILSON-FISCHER FIXED POINT UP TO VERY HIGH ORDER IN ϵ-EXPANSION, WHERE $\epsilon = 4 - d$.

FIXED POINTS OF 4D QFT

THE WISLON-FISCHER FIXED POINT IS NONTRIVIAL IN $d < 4$, BUT IN $d = 4$ SCALAR THEORIES ARE TRIVIAL.

WHAT DO WE KNOW ABOUT OTHER INTERESTING PERTURBATIVELY RENORMALIZABLE $4D$ THEORIES?

- YUKAWA THEORY (INTERACTING SCALARS AND FERMIONS) - TRIVIAL WITH LANDAU POLE
- ABELIAN GAUGE THEORY - QED- TRIVIAL WITH LANDAU POLE
- NONABELIAN GAUGE THEORIES (QCD, STANDARD MODEL) - ASYMPTOTICALLY FREE, I.E. TRIVIAL IN UV, BUT WITH (PERTURBATIVE) LANDAU POLE AT SOME LOW SCALE - Λ_{QCD}

QCD

QCD DESERVES A LITTLE BIT MORE ATTENTION:

$$S = \int d^4x - \frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi}_i \not{D} \psi_i; \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial^a_{\nu\mu} - gf^{abc} A^b_\mu A^c_\nu$$

IT IS CUSTOMARY TO USE INSTEAD OF g, AS THE COUPLING $\alpha_s = \frac{g^2}{4\pi}$.

AT ONE LOOP THE β-FUNCTION IS

$$\frac{\partial \alpha_s}{\partial t} = \beta(\alpha_s) = -\frac{1}{3}(11N_c - 2N_f)\frac{\alpha_s^2}{4\pi}$$

FOR $N_f < \frac{11N_c}{2} = 17$, THIS IS NEGATIVE.

$$\alpha_s(t) = \frac{\alpha_s(t_0)}{1 + \frac{1}{3}(11N_c - 2N_f)\frac{\alpha_s(t_0)}{4\pi}(t - t_0)}$$

THE SITUATION IS OPPOSITE TO QED, OR SCALAR THEORY. GIVEN FINITE $\alpha_s(t_0)$ AT SCALE t_0 NOTHING BAD HAPPENS IN THE UV: $t - t_0 \to \infty$, $\alpha_s(t) \to 0$.

ASYMPTOTIC FREEDOM: QCD IS FREE IN THE UTRAVIOLET.

THERE ARE FORMAL PROBLEMS IN THE IR, THE COUPLING DIVERGES AT $t - t_0 = -\frac{1}{\frac{1}{3}(11N_c - 2N_f)\frac{\alpha_s(t_0)}{4\pi}}$,

$$\Lambda_{QCD} = \mu_0 \exp\{-\frac{12\pi}{(11N_c - 2N_f)}\frac{1}{\alpha_s(t_0)}\}$$

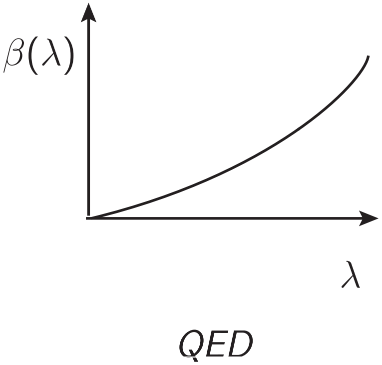
"INFRARED SLAVERY" - THE INTERACTION BECOMES LARGE (INFINITE?) AT LOW MOMENTA - LARGE DISTANCE.

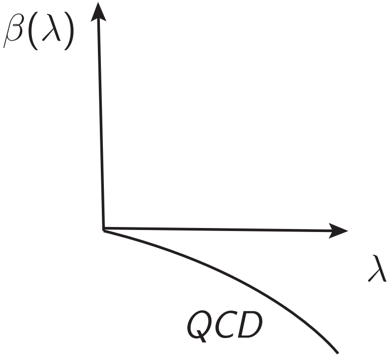
PERTURBATIVE PRECUSRSOR OF CONFINEMENT.

THIS DOES NOT MEAN THAT THE THEORY IS SICK.

THIS MEANS THAT PERTURBATION THEORY CEASES TO WORK AT LOW MOMENTA.

IT IS NOT SO CLEAR THAT THE NOTION OF α_s MAKES SENSE AT LOW MOMENTA: WE DO NOT EXPECT THE LOW ENERGY ACTION TO LOOK ANYTHING LIKE THE ORIGINAL QCD ACTION AT SCALES OF ORDER Λ_{QCD}.


$\beta(\lambda)$
λ
QED


$\beta(\lambda)$
λ
QCD

BANKS-ZAKS FIXED POINT

THERE ARE MORE COMPLICATED THEORIES WHICH EXHIBIT NONTRIVIAL CRITICAL BEHAVIOR IN 4d.
PERTURBATIVE β FUNCTION OF QCD WITH N_c COLORS AND N_f FLAVORS TO TWO LOOPS:

$$\beta(\alpha_s) = -b_0\alpha_s^2 + b_1\alpha_s^3$$

$$b_0 = \frac{1}{4\pi}\frac{1}{3}(11N_c - 2N_f); \qquad b_1 = -\frac{1}{16\pi^2}\left(\frac{34}{3}N_c^2 - \frac{1}{2}N_f\left(2\frac{N_c^2-1}{N_c} + \frac{20}{3}N_c\right)\right)$$

FOR $\frac{11}{2}N_c > N_f > \frac{68N_c^2}{(16+20N_c)}$- AN INFRARED FIXED POINT.

THIS CALCULATION IS PERTURBATIVE, BUT THERE ARE ARGUMENTS BASED ON THE SEIBERG DUALITY THAT THE EFFECT IS REAL.

"CONFORMAL WINDOW" $\frac{1}{3}N_f < N_c < \frac{2}{3}N_f$ IN THE $N = 1$ SUPERSYMMETRIC THEORY - A NONTRIVIAL IR CONFORMAL FIXED POINT.

ANOTHER INTERESTING EXAMPLE IS $N = 4$ SUPER YANG-MILLS THEORY. THE COUPLING CONSTANT IS A GENUINELY MARGINAL OPERATOR (NEITHER MARGINALLY RELEVANT, NOR MARGINALLY IRRELEVANT). FOR ANY VALUE OF λ THE THEORY IS SCALE INVARIANT AND NONTRIVIAL, AND THEORIES WITH DIFFERENT λ ARE DIFFERENT.

ASYMPTOTIC SAFETY

THERE ARE KNOWN EXAMPLES (MOSTLY IN LOWER DIMENSIONS) WHEN PERTURBATIVELY NONRENORMALIZABLE (TRIVIAL) THEORIES, NONPERTURBATIVELY HAVE A FIXED POINT.

THIS FIXED POINT CORRESPONDS TO A STRONGLY INTERACTING QFT AND CANNOT BE SEEN IN PERTURBATION THEORY.

THIS CAME TO BE KNOWN AS "ASYMPTOTIC SAFETY"

ONE KNOWN EXAMPLE: THE GROSS NEVEU MODEL IN 2+1 DIMENSIONS

$$S = \int d^3x \left[\bar{\psi}\partial\!\!\!/\psi - g(\bar{\psi}\psi)^2\right]$$

IN 2+1 D THE DIMENSION OF THE FERMION FIELD IS $[\psi] = 1$, AND SO $[g] = -1$.

ACCORDING TO OUR PERTURBATIVE CRITERION g IS AN IRRELEVANT COUPLING. SO THE THEORY IS PERTURBATIVELY NONRENORMALIZABLE AND CANNOT BE DEFINED AT INFINITE CUTOFF.

NEVERTHELESS ONE FINDS **NONPERTURBATIVELY** THAT

$$g = g_c = \frac{1}{\Lambda}$$

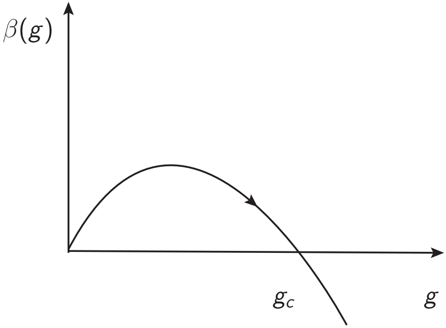
IS A UV FIXED POINT OF THE RG FLOW.

THE COUPLING $M = \frac{1}{g} - \frac{1}{g_c}$ IS THE ONLY RELEVANT COUPLING AROUND THIS NONTRIVIAL FIXED POINT.

FOR $M > 0$ THE IR FIXED IS A FREE MASSLESS FERMION;

FOR $M < 0$, THE CHIRAL SYMMETRY IS SPONTANEOUSLY BROKEN, AND THE FERMION IS MASSLESS IN THE IR.

THE FIXED POINT CORRESPONDS TO THE THE SECOND ORDER CRITICAL POINT OF CHIRAL SYMMETRY BREAKING.



GRAVITY

WEINBERG CONJECTURED THAT EINSTEIN GRAVITY IS ALSO A THEORY OF THIS TYPE.

PERTURBATIVELY GENERAL RELATIVITY IS NONRENORMALIZABLE- - NEWTON'S CONSTANT HAS DIMENSION OF NEGATIVE MASS.

WITHIN PERTURBATION THEORY QUANTUM GRAVITY CANNOT BE DEFINED.

IT IS A TANTALIZING POSSIBILITY THAT GRAVITY IS ASYMPTOTICALLY SAFE, AND THEREFORE CURES IT'S OWN PATHOLOGY.

NEED RELIABLE NONPERTURBATIVE METHODS TO STUDY. "EXACT RENORMALIZATION GROUP" IS ONE SET OF ATTEMPTS.

THE HIGGS NATURALNESS PROBLEM

FINALLY, LET US DISCUSS ONE MORE TOPIC WHERE THE RG SETUP IS USEFUL.

THE ISSUE IS, WHAT IS CALLED "THE NATURALNESS PROBLEM OF LIGHT HIGGS". RECALL - HIGGS IS A SCALAR PARTICLE WITH MASS, WHICH WE

NOW KNOW TO BE AROUND $125 Gev$.
THIS IS AN UNNATURALLY SMALL MASS.
RECALL OUR DISCUSSION OF RUNNING PARAMETERS. OUR RG EQUATION FOR THE RUNNING MASS CAN BE WRITTEN IN THE FOLLOWING FORM

$$\frac{M^2(\mu)}{\mu^2} = \frac{M^2(\mu_0)}{\mu_0^2} \left(\frac{\mu_0}{\mu}\right)^{2-\gamma_{\phi^2}}$$

THE CONSTANT γ_{ϕ^2} IS THE ANALOG OF THE β-FUNCTION FOR COUPLING. IN A WEAKLY COUPLED THEORY $\gamma_{\phi^2} \ll 1$.

LET US TAKE μ_0 AS THE HIGHEST SCALE AT WHICH THE INTERACTION IS WEAK, AND μ AS A LOWER SCALE AT WHICH WE ARE EXAMINING PHYSICS.
IN A WAY μ_0 IS LIKE THE UV CUTOFF ON OUR THEORY; AND IS THE BARE MASS AT THE CUTOFF SCALE.

THE ONLY NATURAL SCALE FOR $M^2(\mu_0)$ IS μ_0 ITSELF.

IN OTHER WORDS THE NATURAL EXPECTATION IS $M^2(\mu_0) = \xi\mu_0^2$.
WHICH THEN GIVES

$$M^2(\mu) = \xi \left(\frac{\mu_0}{\mu}\right)^{2-\gamma_{\phi^2}} \mu^2$$

LET US APPLY THIS RELATION FOR THE STANDARD MODEL WITH HIGGS. CHOOSE $\mu \approx 100\ Gev$, WHICH IS THE SCALE OF WEAK INTERACTIONS. SINCE M_{Higgs} IS ITSELF OF THE ORDER μ, THIS TELLS US THAT

$$\xi \left(\frac{\mu_0}{\mu}\right)^{2-\gamma_{\phi^2}} \sim 1$$

THIS LEAVES US WITH TWO OPTIONS:

- $\mu_0/\mu \sim 1$ - THE NATURAL OPTION. MEANS THAT THE SCALE WHERE SOME NEW STRONG INTERACTION APPEARS IS NOT TOO FAR ABOVE THE WEAK INTERACTION SCALE.
- $\xi \ll 1$ - UNNATURAL OPTION. MEANS THAT THE BARE MASS HAS TO BE TAKEN UNNATURALLY SMALL.

TECHNICOLOR (COMPOSITE HIGGS) IS A NATURAL WAY OUT:

HIGGS IS COMPOSITE; THE THEORY BECOMES STRONGLY INTERACTING AT THE COMPOSITENESS SCALE OF ~ 1 *Tev*.

LARGE ANOMALOUS DIMENSION γ_{ϕ^2} SLOWS DOWN THE RUNNING OF M^2.

ANOTHER OPTION: IMPOSE A SYMMETRY THAT FORCES ξ TO BE SMALL.

SUPERSYMMETRY IS THIS TYPE OF SOLUTION: IT ENSURES $\xi \ll 1$ STILL KEEPING THE THEORY WEAKLY INTERACTING, AND THUS ANOMALOUS DIMENSION SMALL.

THIS IS WHY WE WOULD LIKE TO SEE NEW PHYSICS TO APPEAR NOT FAR ABOVE 1 *Tev*!