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Transmitted ions and ion heating in nearly perpendicular low-Mach number shocks

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Abstract. Nonadiabatic ion heating in low-Mach number shocks is only partially due to reflected ions. Directly transmitted ions contribute significantly into the downstream ion temperature and can be responsible for the whole heating even in the absence of reflected ions, due to insufficient and inhomogeneous deceleration in the cross-shock potential. As a result, the average ion velocity at the downstream edge of the shock ramp is significantly greater than the velocity required by the Rankine-Hugoniot relations, and the ion distribution gyrates as a whole. Because of the nonlinear dependence of the deceleration on the cross-shock potential and initial ion velocity, the gyrating ion distribution is also much more dispersed than the upstream distribution. Additional dispersion is caused by the increase of the vector potential across the shock ramp. The heating depends not only on the bulk shock parameters, as Mach number and β, but also on the field profile. The ion distribution which leaves the ramp is gyrophase-bunched. Further downstream, strong spatially periodic heating occurs, because the initially gyrophase bunched ions become periodically gyrophase dispersed due to nonlinear dependence of the ion gyrophase on its coordinate and velocity.

1. Introduction

It is well-known that ions are strongly heated in supercritical shocks due to reflected ions which are transmitted downstream with high gyration velocities and form strong gyrophase-bunched components in the ion distribution [**?**]. The fraction of the reflected ions can be as large as $\sim 30\%$ in the high-Mach number shocks with $M \sim 10$. Ion heating in low-Mach number shocks is substantially weaker but usually exceeds the adiabatic expectations where $T_\perp \propto B$ [**?**]. Ion dynamics at the shock front is always strongly nonadiabatic since the shock width $\sim c/\omega_{pi} < V_u/\Omega_u$ [**?**]. Here V_u is the upstream plasma velocity, $\Omega_u = eB_u/m_i$ is the upstream ion gyrofrequency, ω_{pi} is the ion plasma frequency, and c is the light velocity. Observations [**??**] have shown that the main ion heating occurs in the shock ramp vicinity (the most narrow part of the shock stationary structure, where the main magnetic field jump occurs), while additional ion distribution broadening and smoothing are observed farther downstream [**?**]. The primary heating occurs during the shock ramp crossing, that is, at the typical timescale of $\lesssim 0.1$ of ion gyroperiod. At the same time the physically meaningful scattering times due to the interaction with the turbulent fields in the shock front constitute several ion gyroperiods [**?**]. Therefore we may ignore the turbulent effects in the shock ramp vicinity and relate the heating to the operation of the quasi-stationary electric and magnetic fields of the shock transition layer. **?** showed that in the shock ramp vicinity the shock structure is quasi-stationary (at least at the ramp crossing timescale) and one-dimensional (the field variation scale in the direction perpendicular to the shock normal is at least an order of magnitude larger than along the shock normal). Since during the shock ramp crossing an ion walks to the distance $\lesssim v_T/\Omega_u$ along the shock front (v_T is the incident ion thermal velocity) and $v_T/V_u \ll 1$, we may also neglect the small deviation of one dimensionality. Thus the prompt main ion heating should occur during the shock ramp crossing because of the interaction with the stationary one-dimensional field structure of the shock ramp.

This idea has been successfully exploited for high-Mach number supercritical shocks where the model of the specular reflection has been applied to estimate the downstream ion temperature at least by an order of magnitude [**??????**]. It was confirmed also by direct hybrid simulations [**???**]

which showed that the main contribution to the downstream ion temperature in supercritical shocks comes from the ions which are reflected and transmitted downstream at their second encounter with the shock ramp. These ions form a very strong gyrophase-bunched (but rather diffuse and not cold) component in the ion distribution in the downstream region. Observations [**??**], simulations [**?**], and analytical models [**?**] show that due to these reflected ions the strong ion heating begins not at the shock ramp but at the upstream edge of the foot, where these ions first appear.

Ion heating in the low-Mach number (subcritical, marginally critical, and supercritical) shocks looks different [**??**]. The heating starts at the shock ramp and not in the foot (the latter does not exist at all in subcritical shocks). The temperature increase is more modest. There are few [**?**] or almost no [**?**] gyrophase-bunched ions in the downstream distribution. Observations are contradictory in some sense, since **?** found that even a small reflected ion fraction of $\sim 3\%$ contributes up to $\sim 40\%$ to the total heating, while **?** found earlier that the contribution of reflected ions both in the number density and temperature is negligible. In both cases the parameters of the observed shocks (Mach number $M \sim 2$, angle $\theta \sim 80^\circ$, and $\beta_e \sim \beta_i \sim 0.1$) were alike, which caused **?** to suggest that the parameter space which determines the shock structure and ion dynamics may be wider than is usually assumed. In all cases it was found that the contribution of the directly transmitted (that is, non reflected) ions in the eventual temperature was significant (decisive in the case of **?**).

? made an attempt to explain the contribution of transmitted ions assuming that they are not deflected when crossing the shock transition layer (which is the ramp for low-Mach number shocks). In the de Hoffman-Teller frame the incident ion velocity is directed along the upstream magnetic field. Since the magnetic field rotates when the shock is crossed, the downstream ion will have a significant gyration velocity component. Heating itself is due to different character of trajectories of different ions. Later hybrid simulations by **?** have shown that the number of reflected ions and their contribution are negligible in the low-Mach number case but drastically increase when the Mach number exceeds the critical Mach number. Still these hybrid simulations did not confirm directly the assumption of ion non deflection. **?** performed similar hybrid simulations to study the dependence of the reflected ion contribution on the temperature tensor. He also criticized the above assumption and argued in favor of what he has called “kinetic effects”. Still the mechanism of the transmitted ion heating remained obscured and the influence of the field structure has not been analyzed.

The objective of the present paper is to study the mechanism of the transmitted ion heating and to assess to what extent these transmitted ions can contribute to the total heating of the downstream ions. The consideration is based on the idea that the heating is mainly due to the nonadiabatic ion motion in the quasi-stationary electromagnetic fields of the shock transition layer. We analyze how the field structure translates into the downstream ion temperature. The consideration is carried out in the perpendicular geometry for simplicity and more transparency. There are arguments [**??**] in favor of applicability of the results for nearly perpendicular shocks as well (see conclusions).

The paper is organized as follows. In section 2 we consider analytically the ion dynamics in the shock front and derive general relations, including ion heating, for low-beta, low-Mach number shocks. In section 3 we illustrate the results of section 2 by direct numerical analysis of ion trajectories in a model shock front. Although the obvious disadvantage of such a test particle analysis is that is does not describe the shock structure self-consistently, it has the following advantages over hybrid simulations: (1) the shock parameters can be freely varied over a wide range, (2) once the field structure is established, the ion motion is governed by the stationary fields, and (3) there are no irreversible processes that can obscure reversible effects.

2. Ion Dynamics in the Shock Front and Downstream Distribution: Analytical Approach

Ion heating in a shock front is mainly a result of the ion dynamics in the spatially varying but time-stationary electric and magnetic fields [**????**]. Although turbulent effects cannot be excluded completely, the level of turbulence is too low to produce the observed heating in such a small crossing time as ~ 0.1 of ion gyroperiod [cf. **?**]. We shall therefore consider a time-stationary one-dimensional shock structure [cf. **?**]. Three dimensionality of real shocks that should be taken into account at larger scales [**?**] (when ion acceleration and/or injection is studied) is unimportant here because the process under consideration is much faster than the cross-field diffusion and develops at much smaller spatial scales.

In the present paper we restrict ourselves to the strictly perpendicular geometry, where we can ignore one degree of freedom. Generalization onto quasi-perpendicular case is discussed in the end of the paper. The shock normal is along x axis, and the magnetic field is along z axis. Since the z motion is decoupled and unaffected by the shock, we restrict ourselves with a two-dimensional problem. In the low-Mach number shocks, there is no or almost no foot, the noncoplanar magnetic field and potential cross-shock electric field are concentrated within the ramp, and the overshoot is low [**??**]. We therefore neglect the overshoot and foot and assume that all field variations occur within the finite shock ramp $-D < x < D$ with the length $L = 2D$. The fields in the shock are $\mathbf{B} = [0, 0, B_z(x)]$, with $B_z = B_u$ at $x < -D$, $\mathbf{E} = [E_x(x), E_y, 0]$, where $E_y = V_u B_u = \text{const}$ and $E_x < 0$.

Notation

As we shall see below the problem is essentially dimensionless, so it is natural to express all quantities in the corresponding dimensionless variables as follows:

$$\frac{\mathbf{v}}{V_u} \to \mathbf{u}, \quad \Omega_u t \to t, \quad \frac{\Omega_u x}{V_u} \to x, \tag{1}$$

where we retained the same notation for dimensional and corresponding dimensionless variables for convenience. The normalized magnetic field b, electric field e, and potential ϕ are defined as follows:

$$\frac{B_z}{B_u} = b, \quad \frac{E_x}{E_y} = e, \quad \frac{e\varphi}{m_i V_u^2/2} = \phi, \tag{2}$$

where $E_x = -d\varphi/dx$ takes the following dimensionless form $e = -(1/2)(d\phi/dx)$.

In what follows we use the dimensionless upstream ion thermal velocity $v_T = (T_u/m_i)^{1/2}/V_u = \sqrt{\beta_i/2M^2}$, where $\beta_i = 8\pi n_u T_u/B_u^2$ and M is the Alfven Mach number. We normalize the temperature with the upstream temperature $T/T_u \to T$ (assuming Maxwellian distribution for the incident ions) and the pressure with the incident ion ram pressure $p_{ij}/n_u m_i V_u^2 \to p_{ij}$.

In the new dimensionless variables the ion equations of motion take the following simple form

$$\dot{u}_x = e + u_y b, \quad \dot{x} = u_x, \tag{3}$$

$$\dot{u}_y = 1 - u_x b, \quad \dot{y} = u_y. \tag{4}$$

We shall denote also $\mathbf{v} = \mathbf{u}$ at the upstream edge of the ramp $x = -D$ and $\mathbf{w} = \mathbf{u}$ at the downstream edge of the ramp $x = D$.

Collisionless Distribution and Moments

Since ions are assumed to be collisionless, the distribution function $f(u_x, u_y, x) = f_0(u_{x0}, u_{y0}, x_0)$, where (u_{x0}, u_{y0}, x_0) are the initial conditions for the ion trajectory $(u_x, u_y, x)(t)$ found as a solution of (**??**)–(**??**). In what follows we put $x_0 = -D$, so that $(u_{x0}, u_{y0}) = (v_x, v_y)$. Then

$$f(u_x, u_y, x) = f_0[v_x(u_x, u_y, x), v_y(u_x, u_y, x)]. \tag{5}$$

Averaging of any function $g(u_x, u_y)$ over the distribution $f(u_x, u_y)$ can be carried out as follows

$$\begin{aligned}\langle g\rangle &= \int g(u_x, u_y) f(u_x, u_y) du_x du_y \\ &= \sum_s \int g(u_x, u_y) J f_0(v_x, v_y) dv_x dv_y \\ &= \sum_s \int g(u_x, u_y) |v_x|/|u_x| f_0(v_x, v_y) dv_x dv_y,\end{aligned} \tag{6}$$

where $J = \det ||\partial u_i/\partial v_j||$ is the Jacobian of the transformation $\mathbf{v} \to \mathbf{u}$ at $x = \text{const}$ and the last equality follows from the general relation $J = |v_x|/|u_x|$ which is valid in the one-dimensional stationary case. In (**??**), now u_x and u_y are functions of (v_x, v_y, x) and summation over s takes into account the possibility that there is more than one solution.

The moments of the distribution are found now as follows

$$n = \langle 1\rangle, \quad V_i = \langle u_i\rangle/n, \tag{7}$$

$$p_{ij} = \langle u_i u_j\rangle - n V_i V_j, \tag{8}$$

$$T_{ij} = p_{ij}/n, \tag{9}$$

where n, V_i, p_{ij}, and T_{ij} are density, velocity, pressure tensor, and temperature tensor, respectively.

When the distribution is warm (not hot) everywhere, that is, $v_T \ll V$, we Taylor expand $\mathbf{u} = \mathbf{u}_c + \delta\mathbf{u}$ and $\mathbf{v} = \mathbf{v}_c + \delta\mathbf{v}$ near the trajectory of the central ion $\mathbf{u} = \mathbf{u}_c(\mathbf{v}_c, x)$. Then simple but somewhat tedious algebra gives

$$n = |v_{xc}|/|u_{xc}|, \qquad V_i = u_{ic}, \tag{10}$$

$$T_{ij}| = \frac{\partial u_i}{\partial v_l} || \frac{\partial u_j}{\partial v_m} | T_{lm,0}, \qquad p_{ij} = n T_{ij} \tag{11}$$

where summation over repeated indices is understood and the derivatives are taken at $\mathbf{u} = \mathbf{u}_c$.

Shock Ramp Crossing

Let us consider first the shock ramp crossing, assuming that the distribution function $f_0(v_x, v_y)$ of the incident ions at the upstream edge of the shock ramp $x = -D$ is known. We assume that the ions do not return to the ramp and cross it again from downstream to upstream. This assumption is applied to the whole distribution for low-Mach number shocks [**?????**] and to the directly transmitted part of the ion distribution in high-Mach number shocks [**???**]. Taking into account that the directly transmitted ions do not stop inside the shock ramp, that is, $u_x > 0$ in the region $-D < x < D$, one can substitute $d/dt = u_x\, d/dx$, and (**??**)-(**??**) take the following form:

$$u_x^2 = v_x^2 - \phi + 2\int_{-D}^{x} u_y b dx, \qquad (12)$$

$$u_y = v_y + \int_{-D}^{x} (1/u_x - b) dx, \qquad (13)$$

where it is assumed that $\phi(x = -D) = 0$.

The ion velocities (w_x, w_y) at the downstream edge of the shock ramp are (formally) obtained from (**??**) and (**??**) by putting the upper integration level at $x = D$. Equations (**??**)-(**??**) are not integrable and cannot be solved in general case. We solve them in the low-β limit $v_T \ll 1$, assuming that the terms containing integration can be treated as small. Then in the first order one has

$$w_y = v_y + \int_{-D}^{D} [(1-\phi)^{-1/2} - b] dx, \qquad (14)$$

$$w_x^2 = v_x^2 - \phi_0 \qquad (15)$$
$$+ 2\int_{-D}^{D} b dx \int_{-D}^{x} [(1-\phi)^{-1/2} - b] d\xi,$$

where $\phi_0 = \phi(x = D)$. Equation (**??**) gives the condition of the applicability of the perturbative approach as

$$v_T \ll (1-\phi_0)^{1/2},$$
$$|\int_{-D}^{D} b dx \int_{-D}^{x} [(1-\phi)^{-1/2} - b] d\xi| \ll (1-\phi_0)^{1/2}. \qquad (16)$$

For the low-beta Maxwellian distribution of the incident ions (centered on $v_x = 1$, $v_y = 0$), (**??**)–(**??**) with (**??**)–(**??**) and (**??**)–(**??**) give for the variables at the downstream edge of the shock ramp (superscript r)

$$n^{(r)} = \{1 - \phi + 2\int_{-D}^{D} b dx$$
$$\cdot \int_{-D}^{x} d\xi [(1-\phi)^{-1/2} - b]\}^{-1/2}, \qquad (17)$$

$$V_x^{(r)} = 1/n^{(r)}, \qquad (18)$$

$$V_y^{(r)} = \int_{-D}^{D} dx [(1-\phi)^{-1/2} - b], \qquad (19)$$

$$T_{xx}^{(r)} = (n^{(r)})^2 \{[1 - \tfrac{1}{2}\int_{-D}^{D} b dx$$
$$\cdot \int_{-D}^{x} d\xi (1-\phi)^{-3/2}]^2 + (\int_{-D}^{D} b dx)^2\}, \qquad (20)$$

$$T_{yy}^{(r)} = 1 + \tfrac{1}{4}[\int_{-D}^{D} dx (1-\phi)^{-3/2}]^2, \qquad (21)$$

$$T_{xy}^{(r)} = n^{(r)} \{1 - \tfrac{1}{2}\int_{-D}^{D} b dx \int_{-D}^{x} d\xi (1-\phi)^{-3/2}$$
$$- \tfrac{1}{2}(\int_{-D}^{D} b dx)[\int_{-D}^{D} dx (1-\phi)^{-3/2}]\}, \qquad (22)$$

and the pressure tensor $p_{ij}^{(r)} = n^{(r)} T_{ij}^{(r)}$. One can see that although the incident ion temperature is diagonal, there is a nonzero off-diagonal T_{xy} at the downstream edge of the shock ramp.

As can be seen from (**??**) to (**??**) the heating at the ramp depends on the profiles of the magnetic and electric field and not only on the upstream and downstream values. This dependence is weak when the shock is not thick. In the limit of the thin shock $D \to 0$ one immediately finds that the heating depends only on the cross-shock potential:

$$n^{(r)} \to 1/V_x^{(r)} \to (1-\phi)^{1/2}, \quad V_y^{(r)} \to 0, \qquad (23)$$

$$T_{xx}^{(r)} \to (1-\phi), \quad T_{yy}^{(r)} \to 1, \quad T_{xy}^{(r)} \to (1-\phi)^{1/2}. \qquad (24)$$

One can expect that in the low-Mach number shocks, where the shock width $\sim D \sim 1/M$ [**?**], the heating at the ramp is close to that described by (**??**)–(**??**), with relative corrections $\lesssim D$.

Proceeding Further Downstream

In the downstream region, $E_x = 0$ and $B = B_d = RB_u$, where $R =$ const, so that the corresponding equations of motion reduce to

$$\dot{u}_x = R u_y, \qquad \dot{x} = u_x, \qquad (25)$$

$$\dot{u}_y = R(\lambda - u_x), \qquad \dot{y} = u_y, \qquad (26)$$

where we introduced $\lambda = 1/R$. It is convenient to use the ion velocities (w_x, w_y) at the downstream edge of the shock ramp $x = D$ as new initial conditions. Then the corresponding solution of (**??**)–(**??**) reads

$$u_x = \lambda + (w_x - \lambda)\cos\psi + w_y \sin\psi, \quad (27)$$

$$u_y = -(w_x - \lambda)\sin\psi + w_y\cos\psi, \quad (28)$$

$$Rx = \lambda\psi + (w_x - \lambda)\sin\psi + w_y\cos\psi, \quad (29)$$

where $\psi = Rt$ is the ion gyrophase.

In principle (but unfortunately not in practice), (**??**)–(**??**) allow us to exclude u_x and u_y and find ψ as a function of initial velocities and x. We shall therefore define (multi valued) gyration function $\psi_s(w_x, w_y, x)$ and rewrite the averaging (**??**) as follows

$$\langle g(u_x, u_y)\rangle_d = \sum_s \int f_r(w_x, w_y) dw_x dw_y \frac{|w_x|}{|u_x^{(s)}|} g, \quad (30)$$

where subscripts d and r denote downstream and downstream edge of the shock ramp, respectively, and the summation is over all solutions (u_x, u_y) for given (w_x, w_y, x).

In the same approximation of the warm ion distribution as above one finds the following parametric representation

$$n = 1/u_x = [\lambda + (w_x - \lambda)\cos\psi + w_y\sin\psi]^{-1}, \quad (31)$$

$$\begin{aligned} T_{xx}^{(d)} &= \{[\lambda\cos\psi + (w_x - \lambda)]^2 T_{xx}^{(r)} \\ &+ (w_y + w_x\sin\psi)^2 T_{yy}^{(r)} \\ &+ 2[\lambda\cos\psi + (w_x - \lambda)](w_y + w_x\sin\psi)T_{xy}^{(r)}\}/u_x^2, \end{aligned} \quad (32)$$

$$\begin{aligned} T_{yy}^{(d)} &= [\lambda^2\sin^2\psi T_{xx}^{(r)} \\ &+ (w_x\cos\psi + w_y\sin\psi)^2 T_{yy}^{(r)} \\ &- 2\lambda\sin\psi(w_x\cos\psi + w_y\sin\psi)T_{xy}^{(r)}]/u_x^2, \end{aligned} \quad (33)$$

$$\begin{aligned} T_{xy}^{(d)} &= \big((- \lambda\sin\psi[\lambda\cos\psi + (w_x - \lambda)]T_{xx}^{(r)} \\ &+ (w_y + w_x\sin\psi)(w_x\cos\psi + w_y\sin\psi)T_{yy}^{(r)} \\ &+ \{[\lambda\cos\psi + (w_x - \lambda)](w_x\cos\psi + w_y\sin\psi) \\ &- \lambda\sin\psi(w_y + w_x\sin\psi)\}T_{xy}^{(r)})\big)/u_x^2, \end{aligned} \quad (34)$$

where u_x and x are given by (**??**) and (**??**), while $\mathbf{w} = (w_x, w_y)$ is the initial velocity of the central ion. The pressure components, as above, are related to the temperature tensor as $p_{ij} = nT_{ij}$. One can immediately see that the temperature tensor (as well as density and velocity) is spatially periodic with the period $2\pi\lambda^2$. The last conclusion is valid for arbitrary initial distribution $f_r(w_x, w_y)$ of the ions at the downstream edge of the shock ramp and not only in the warm case. If for each ion, $(w_x - \lambda)^2 + w_y^2 < \lambda^2$ (that is, $u_x > 0$ always), the periodicity starts at the very edge of the ramp. Ions, trajectories of which are looped $(w_x - \lambda)^2 + w_y^2 > \lambda^2$, break this periodicity near the ramp, and it starts at some distance farther into downstream.

In the same thin shock approximation as above one has $w_x = \sqrt{1-\phi}$, $w_y = 0$, $T_{xx}^{(r)} = 1/(1-\phi)$, $T_{xy}^{(r)} = 1/\sqrt{1-\phi}$, $T_{yy}^{(r)} = 1$, and since $\lambda = 1/R = B_u/B_d$, the resulting heating is function of ϕ and R only. Analysis of (**??**)–(**??**) shows that $T = (T_{xx} + T_{yy})/2$ reaches its maximum approximately at $\cos\psi = -1$, that is, where n and T_{yy} are maximum. The maximum temperature at this point is

$$T_{\max}^{(d)} \approx \frac{R^2(1-\phi)}{2[2 - R(1-\phi)^{1/2}]^2}, \quad (35)$$

and depends on the cross-shock potential and magnetic compression ratio. However, the above warm ion approximation works at $\cos\psi = -1$, if the corresponding thermal velocity does not exceed $V_{x,\min} = 2\lambda - \sqrt{1-\phi}$, which gives independent estimate of the upper limit

$$T_{\max}^{(d')} \approx \frac{2M^2}{\beta_i}(2\lambda - \sqrt{1-\phi})^2. \quad (36)$$

The correct estimate is then the smallest from (**??**) and (**??**).

Summarizing the above consideration, the ion distribution at the downstream edge of the shock ramp is determined by the nonadiabatic ion motion in the stationary electric and magnetic fields of the ramp. The cross-shock potential is not sufficient to decelerate ions down to the velocity required by the Rankine-Hugoniot relations, which results in the gyration as a whole of the downstream ion distribution. As a result, all hydrodynamical variables become spatially periodic (oscillate with x) downstream of the ramp, in the near vicinity of it, where turbulent effects do not isotropize the distribution yet. Nonlinear dependence of the ion deceleration and magnetic deflection on the initial ion velocity results in the increase of the velocity dispersion in the ion distribution at the downstream edge of the ramp and therefore nonadiabatic heating. This heating depends on the profiles of the electric and magnetic field (in particular, shock width) and not only on the upstream and downstream values. Further downstream the effective heating is due to the gyration of the ion distribution and resulting spatially periodic broadening of ion distribution, because of the nonlinear dependence of the ion gyrophase on its position and velocity. The maximum possible temperature is determined mainly by the excess of gyration energy of ions, which in turn depends mainly on ϕ and R. Therefore downstream outside of the ramp ion heating should be much less sensitive to the field profile than at the ramp itself. It should be emphasized that in all cases the heating is due to insufficient deceleration of ions at the shock ramp.

3. Numerical Analysis

As we have seen earlier, complete analytical consideration is possible only for low-beta cases. Mathematical diffi-