Dissipation and Diffusion of Particles in Random Environments

Thesis submitted in partial fulfillment of the requirement for the degree of doctor of philosophy

Submitted by: **Yoav Etzioni**

Submitted to the senate of Ben Gurion University of the Negev

March 9, 2012

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Adviser's Approval	__________________	Date:	__________________
Kreitman's Dean Approval	__________________	Date:	__________________

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This work was carried out under the supervision of

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The work in this thesis was published in the following papers

- P. Le Doussal, Y. Etzioni and B. Horovitz, J. Stat. Mech. P07012 (2009) [1]
- Y. Etzioni, B. Horovitz and P. Le Doussal, Phys. Rev. Lett. 106, 166803 (2011) [2] (appendix C)
- Y. Etzioni, B. Horovitz and P. Le Doussal, In preparation [3]

Abstract

In this work we study the dynamics of a particle on a ring in presence of a dissipative Caldeira-Leggett (CL) environment. In the first part of the work we study this model using the Keldysh non-equilibrium formalism to derive the particle's response to an external DC field. From the response we find the renormalized dissipation η^R up to second order and using a renormalization group analysis find that for a large dissipation parameter $\eta > \eta_c$ η^R flows to a fixed point $\eta_c = \hbar/2\pi$. We also study the semiclassical limit of the problem where we show that the model reduces to a Langevin equation and study the equation numerically. For the semiclassical limit we also expand the model for a more general environment, that of a dirty metal (DM).

We reexamine the mapping of the CL problem to that of the Coulomb box and find that a certain average of the relaxation resistance is quantized for large η and propose a box experiment to measure the corresponding quantized noise.

In the second part of this work we study equilibrium properties. Using the Matsubara imaginary path integral formalism we analyze the model perturbatively for both large and small η. We develop a Monte Carlo (MC) algorithm to solve this problem. However, when the flux through the ring is half the quantum flux, we encountered the the infamous sign problem, hence our numerical data cannot identify η^R.

Motivated by the small η behaviour of the particle in the classical limit, we consider in the last part of this work the winding angle ϕ_t around the center of a smooth Gaussian process in the plane with arbitrary correlation, where the CL correlation is but one choice for this correlation. We obtain the stationary distribution of $\dot{\phi}_t$ as well as a closed formula as a function of the correlation function for the variance of the winding angle, the correlations of $\mathrm{e}^{in\phi_t}$ with integer n and the variance of the algebraic area enclosed by the process. Those results are tested numerically.

Contents

1 Introduction

Conductance in the context of quantum mechanics is shown to be quantized in various models such as the quantum Hall effect and one dimensional wires. In this work we discuss a new case of conductance quantization for the model of particle on a ring affected by a noisy environment and relate it to the quantization of the relaxation resistance in a single electron box (SEB).

Following the prediction of Büttiker, Thomas and Prêtre [4] the quantization of the relaxation resistance R_q, defined via an AC capacitance of a single electron box (SEB) is of recent interest. The theory has been recently extended to include Coulomb blockade effects [5] showing that $R_q = h/2e^2$ is valid for small dots and crosses over to $R_q = h/e^2$ for large dots. A quantum mesoscopic RC circuit has been implemented in a two-dimensional electron gas [6] and $R_q = h/2e^2$ has been measured.

The problem of a single particle on a ring under the influence of a dissipative environments has been considered in many theoretical past works particularly for studying the renormalization of the mass M^* and its possible relation to dephasing [7–11]. A recent study has observed Aharonov-Bohm oscillations from single electron states in semiconducting rings [12].

The common prototype for the description of a dissipative environment is that of Caldeira-Leggett (CL) [13], where the environment is modeled as a large set of oscillators linearly coupled to a quantum system. In this work we present a detailed study of this model, namely a particle confined to a ring, driven by a tangent electric DC field. The latter is caused by a magnetic flux through the ring linearly changing in time and subjected to a CL environment. A schematic figure of the model is given in Fig. 1.

There is a known mapping between the SEB and the model of a particle on a ring affected by CL environment [14, 15]. While the exact mapping assumes weak tunneling into the box with many channels, it has been extensively used to describe various tunnel junctions [16],

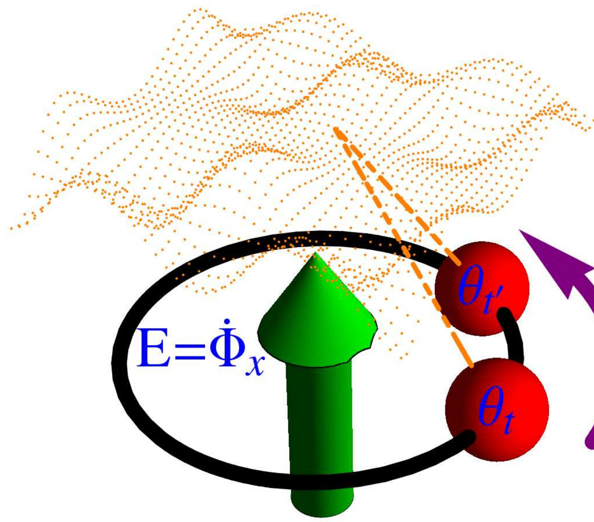

Figure 1: Model of a particle on a ring affected by a noisy environment. A tangent electric field E or an equivalent magnetic flux through the ring that increases linearly with time creates a force on the particle, while a dissipative environment creates noise with long range time correlation and a frictional force on the particle.

the Coulomb blockade phenomenon in single electron box (SEB) and in the single electron transistor (SET) [16–27].

In the main part of this work, section 2 we study this ring model using the Keldysh method for non-equilibrium dynamics and a renormalization group (RG) reasoning. We find that a small parameter in the perturbation theory contains a periodic function of the dissipative parameter η, this parameter is mapped to the lead dot coupling in the SEB. The RG equation suggests that this dissipative parameter flows to a fixed point $\eta = \eta_c$ with $\eta_c = \frac{\hbar}{2\pi}$. We also examine the mapping between the two models and show that a certain average of the relaxation resistance R_q is quantized for finite $\eta > \eta_c$ [2, 3].

A related approach for the study of quantum systems coupled to dissipative environments is the influence functional of Feynman-Vernon [28], which express the influence of the environment on the system. An exponential decay in time of the functional identifies a dephasing time τ_ϕ. Using notations of the present work the attenuation factor is $F_\tau = \mathrm{e}^{-\frac{1}{2}\tilde{C}_\tau}$, which for a dissipative environment at high temperatures is $\sim \mathrm{e}^{-\tau/\tau_\phi}$ and $\tilde{C}_\tau$ is a correlation function of the system to be define later. A study of mesoscopic systems coupled to both an external driving force and to a dissipative environment [29], found that the two effects of the environment, dissipation and dephasing, as described above, are competing effects. The dissipation mechanism in the system corresponds to Landau-Zener transitions through a gap, i.e. avoided crossings, e.g. as for rings with static disorder. Dephasing is the destruction of quantum coherence, the latter being responsible for localization in energy space. The destruction of localization therefore enhances the rate at which energy is pumped into the system, increasing the Landau-Zener transition rate. A competing effect is the relaxation, i.e. the rate at which energy is leaving the system into the environment. Due to interplay between these two effects the conductance depends on the external field in a nonmonotonic way. In a recent work [30] dephasing of the particle in a ring coupled to dissipative environments of either CL or DM types was calculated in small η perturbation at finite temperatures.

In section 3 we study this model in Equilibrium, both perturbatively and using Monte Carlo (MC) simulations and try to show this phase transition. However, we encounter a sign problem, so that the MC results are not conclusive.

In section 4 we study the behavior of an arbitrary correlated noise (or process) on a two dimensional complex plane. For such noise one can define the norm and the phase of that noise in the form $\xi_t = \xi_t^x + i\xi_t^y = |\xi_t|\,\mathrm{e}^{i\phi_t}$ where $\xi_t^{i=x,y}$ are a real Gaussian noise function with arbitrary correlation function which is independent of i. Specifically we ask what is the time correlation function and other related properties of the noise phase ϕ_t and how are these properties depend on the corelation function of ξ_t^i. We give a detailed answer for this question.

This section is motivated by the Langevin equation of section 2. For the case where the ξ_t^i

correlation function is that of CL ϕ_t corresponds to the semiclassical solution of the particle dynamics in the small η limit. This section was published in [1].

A proposed box experiment that verifies the result in section 2 is given with the summary of the work in section 5.

2 Particle on a ring and the coulomb box in dissipative environments

2.1 Introduction

In the present section we address the problem of a particle on a ring affected by a CL environment and find that the dissipation parameter η is quantized. We examine the mapping of the SEB to this problem and relate this quantization to the known quantization of the relaxation resistance R_q in SEB. The essence of this section was published [2] (reprint in appendix C) and future publication is in preparation [3].

We begin in section 2.2 with examining the mapping of the SEB problem and the ring model. In terms of the SEB, our results extend the previous analysis [5] of the relaxation resistance R_q to the case of many channels N_c [31]. We note that for $N_c > 1$ the relaxation resistance for noninteracting electrons becomes $h/(2N_c e^2)$ [4]; no result exists for interacting electrons. We find that for strong coupling, $\eta/\hbar \gtrsim 1$ the relaxation resistance is quantized to e^2/h up to an exponentially small correction $\sim \mathrm{e}^{-\pi\eta/\hbar}$. For finite η, but still $\eta > \eta_c$ we find that a certain average of the relaxation resistance is quantized (see Eq. (2.78) below).

In our approach to the study of the ring problem, we evaluate the response to a strictly DC electric field E, equivalent to a magnetic flux through the ring that increases linearly with time, meaning a non-equilibrium response. We therefore use a real time Keldysh method which is derived in section 2.3. While thermodynamic properties of ring problems has been much studied, including extensive MC studies [19, 20] of M^*, no sign of a finite coupling fixed point has been detected. We claim that thermodynamic quantities like M^*, that are flux sensitive, decouple from the response to electric field E, a response that averages over flux values.

In section 2.4 we consider the semiclassical limit of this action, which is equivalent to the large dissipation limit, and find perturbative results for the Green's function in powers of large η. These results are compared with known results [17, 18] from equilibrium formulation and we discuss a possible reason by which a nonequilibrium formulation of the problem gives different result. In the following section 2.5 we show that the above semiclassical action relates to a Langevin equation, and study the equation using numerical simulations. We discuss limitation of the numerical procedure. In addition to the CL environment we study in section 2.6 the more general case where the environment is not that of CL but rather that produced by a dirty metal, the relevant Langevin equation is derived and some numerical results are shown.

In section 2.7 we derive the large η perturbation for the quantum problem and in 2.8 we address this expansion as a 2-loop RG equation. We find that the perturbation theory identifies an unexpected new small parameter $\sin(\frac{\hbar}{2\eta})$ and infer that a large η flows to the above mentioned fixed point $\eta^R = \eta_c$An intuitive argument for this quantization as well as the conclusions are given in section 2.9.

2.2 Mapping between the particle on a ring and a Coulomb box

In the present section we examine the mapping of the box and ring problems. The action for the SEB is given by

$$S = \int_t \left\{ \sum_{\alpha n} d^\dagger_{\alpha n}(i\hbar\partial_t - \epsilon_\alpha)d_{\alpha n} - E_c(\hat{N} - N_0)^2 \right\} + S_{lead} + S_{tun} \tag{2.1}$$

where $d_{\alpha n}$ are dot electron operators, $n = 1, ..N_c$ labels the channels, $\hat{N} = \sum_{\alpha n} d^\dagger_{\alpha n} d_{\alpha n}$, $E_c = e^2/(2C_g)$ with C_g is the geometric (bare) capacitance, N_0 is proportional to the gate voltage, S_{lead} describes free electrons on the lead and S_{tun} is the tunneling between the lead and the dot. We introduce an auxiliary variable θ_t with an action $E_c \int_t [\hat{N} - N_0 - \hbar\dot{\theta}/2E_c]^2$ and rewrite the total action as

$$S = \int_t \left\{ \sum_{\alpha n} d^\dagger_{\alpha n}(i\hbar\partial_t - \epsilon_\alpha - \hbar\dot{\theta}_t)d_{\alpha n} + \frac{\hbar^2\dot{\theta}_t^2}{4E_c} + N_0\hbar\dot{\theta}_t \right\} + S_{lead} + S_{tun}\,. \tag{2.2}$$

In terms of fermion operators $\tilde{d}_{\alpha n} = \mathrm{e}^{i\theta(t)} d_{\alpha n}$, integrating out these fermions and expanding in S_{tun} yields the well known effective action for the SEB [14–17, 19–27]. Eq. (2.2) shows that the equivalent particle on a ring has a mass $M = \hbar^2/(2E_c)$ (the radius of the ring is chosen as $= 1$) and there is a flux (in unit of the flux quantum) $\phi_x = -N_0$ through the ring. The tunneling amplitudes squared, weighted by the number N_c of channels, become the dissipation parameter η of the particle. The mapping becomes exact in the large N_c limit at fixed η and for small mean level spacing [32] $\Delta \ll E_c$, a situation that can be realized [31]; the application of this mapping is therefore limited to the temperature range $\Delta < T \ll E_c$. Furthermore, by shifting $\hbar\dot{\theta}_t \to \hbar\dot{\theta}_t + 2E_c(\hat{N}_t - N_0)$ we obtain $\hbar\langle\dot{\theta}_t\rangle = 2E_c[\langle\hat{N}\rangle_{N_0} - N_0]$ and also a relation between response functions

$$\hbar^2 \tilde{K}_{t,t'} \;=\; -2E_c\hbar\delta(t-t') + 4E_c^2 K_{t,t'} \tag{2.3}$$

where $\tilde{K}_{t,t'} = +i\theta(t-t')\langle[\dot{\theta}_t, \dot{\theta}_{t'}]\rangle$ is the response for the ring while $K_{t,t'} = +i\theta(t-t')\langle[\hat{N}_t, \hat{N}_{t'}]\rangle$ is for the SEB. The $-2E_c\hbar\delta(t-t')$ term in (2.3) is essential, e.g. without tunneling the charge fluctuations are frozen, $K_{t,t'} = 0$, while the corresponding particle is free with the correlation $-2E_c\hbar\delta(t-t')$.

The SEB response is parameterized as [5] $\frac{e^2}{\hbar}K_\omega = C_0(1 + i\omega C_0 R_q)$ where C_0 is the effective DC capacitance and R_q is the relaxation resistance [4]. The corresponding $\tilde{K}_{t,t'}$ is parameterized as

$$\tilde{K}_\omega = -K_0(\phi_x) + i\omega K_1(\phi_x) + \mathcal{O}\big(\omega^2\big). \tag{2.4}$$

The fluctuation dissipation theorem (FDT) relates $\tilde{K}_\omega$ and the linear response to $\delta\mathcal{H}_{ring} = +\hbar\dot{\theta}\delta\phi_x(t)$

$$\hbar\langle\dot{\theta}_t\rangle = -\int_{t'} \tilde{K}_{t,t'}\delta\phi_x(t') \tag{2.5}$$

The response term K_0 corresponds to the persistent current, i.e. for a time independent flux one can integrate the last expression to get

$$\langle \dot{\theta}_t \rangle = \int_0^{\phi_x} K_0(\phi'_x) d\phi'_x \tag{2.6}$$

The continuation to imaginary time identifies the curvature of the free energy [7–11, 14, 15], or an effective mass, as $\frac{1}{\hbar}\frac{\partial^2 F}{\partial \phi_x^2} = \hbar/M^*(\phi_x) = K_0(\phi_x)$; e.g. without tunneling $M^* = M$ while for large η the effective mass $M^* \sim \mathrm{e}^{\pi\eta/\hbar}$ is exponentially large.

Consider now the system in presence of a (classical) electric field E, of Hamiltonian $\delta\mathcal{H}_{ring} = -(E + \delta E(t))\theta$ and define the linear response $\delta\langle\theta_t\rangle_E = \int_{t'} R_{t,t'}\delta E(t')$ to a small perturbation δE. This response is studied below for a DC field. In general its low frequency form is (see Eq. (2.16) below) $R_\omega = \frac{-1}{i\omega\eta^R(E)}$ which defines $\eta^R(E)$ as a renormalized dissipation parameter. Since $E = \hbar\dot{\phi}_x$ we expect $\hbar\omega^2 R_\omega = \tilde{K}_\omega$, hence the K_0 term in Eq. (2.4) is not reproduced. To resolve this discrepancy we note that an additional constant flux ϕ_x in the total flux $\phi_x + Et/\hbar$ can be eliminated by redefining the origin of the time t, therefore the persistent current part should be eliminated. More precisely, define $\hbar\phi_x(t) = Et$; the 1st term in (2.4) $K_0(\phi_x) = K_0(Et/\hbar)$ becomes a periodic function, i.e. an AC response at $\omega_E = 2\pi E/\hbar$. For a DC response at finite E this persistent current response averages to zero, i.e. $\int_0^1 K_0(\phi_x)d\phi_x = 0$. The same reasoning applies to a ϕ_x average on $K_1(\phi_x)$. Hence the DC response to a DC field is given by

$$\lim_{E\to 0}\lim_{\omega\to 0}\frac{\tilde{K}_\omega}{\omega} = i\int_0^1 K_1(\phi_x)d\phi_x\,. \tag{2.7}$$

Therefore $\hbar/\eta^R = \int_0^1 K_1(\phi_x)d\phi_x$ where we denote $\eta^R \equiv \eta^R(E \to 0)$. The order of limits in (2.7) signifies that η^R is essentially a non-equilibrium response. The physical picture is that in a DC field the particle rotates around the ring and produces two types of currents. First is the persistent current that oscillates in time as ϕ_x increases and is therefore time averaged to zero; this current is non-dissipative. Second, there is a genuine DC response from the $i\omega K_1$ term, which is dissipative.

In terms of the SEB response, using Eq. (2.3), we obtain the following mapping of ring and box parameters as functions of flux ϕ_x and N_0:

$$\begin{aligned} \frac{M}{M^*(\phi_x)} &= 1 - \frac{C_0(N_0)}{C_g} \\ \frac{\hbar}{\eta^R} &= \frac{e^2}{\hbar}\int_0^1 \frac{C_0^2(N_0)}{C_g^2} R_q(N_0) dN_0 \end{aligned} \tag{2.8}$$

and we note also that $\int_0^1 C_0(N_0)dN_0 = C_g$.

2.3 Keldysh path integral formulation for action on a ring

In this section we introduce the model used for the particle on a ring in non-Equilibrium. To derive the Keldysh action [33, 34] , we start from the known partition function of a particle in a CL environment [13]

$$Z = \int_{\mathcal{D}\hat{\mathbf{x}}_t, \mathcal{D}\mathbf{x}_t} \mathrm{e}^{-S_K[\mathbf{x},\hat{\mathbf{x}}]} \tag{2.9}$$

the action S_K in two dimensions with a position vectors $\mathbf{x}^{\pm}$, where $\pm$ correspond to the upper and lower Keldysh contour is

$$S_K[\mathbf{x},\hat{\mathbf{x}}] = i\int_{t,t'} \hat{\mathbf{x}}_t R^{-1}_{t,t'}\mathbf{x}_{t'} + \tfrac{1}{2}\int_{t,t'} \hat{\mathbf{x}}_t B_{t,t'}\hat{\mathbf{x}}_{t'} \tag{2.10}$$

the fields $\mathbf{x}_t = \frac{1}{2}(\mathbf{x}_t^+ + \mathbf{x}_t^-)$ and $\hat{\mathbf{x}}_t = \frac{1}{\hbar}(\mathbf{x}_t^+ - \mathbf{x}_t^-)$ are known as the classical and quantum fields respectively. The retarded and the Keldysh Green's function in time and frequency spaces are

$$\begin{aligned} R_\tau &= \frac{1}{\eta}(1-\mathrm{e}^{-\eta\tau/m})\Theta(\tau) \qquad & R_\omega &= -1/[m\omega^2 + i\eta\omega] \\ B_\tau &= -\hbar\eta/\left(\pi\tau^2\right) \quad \tau \neq 0 \qquad & B_\omega &= \hbar\eta\left|\omega\right|. \end{aligned} \tag{2.11}$$

This quadratic action corresponds to a particle of mass m and a friction parameter η within a Langevin equation

$$M\ddot{\mathbf{x}} + \eta\dot{\mathbf{x}} = \xi_t \tag{2.12}$$

each component of $\xi_t = (\xi_t^x, \xi_t^y)$ is a random number with correlation function $\left\langle |\xi_i|_\omega^2 \right\rangle = B_\omega \delta_{ij}$. We project the position on a 2 dimensional ring by

$$x_t^+ = [\cos\theta_+(t), \sin\theta_+(t)] \qquad x_t^- = [\cos\theta_-(t), \sin\theta_-(t)] \tag{2.13}$$

and implicitly assume $R = 1$ and all parameters are length dimensionless. Writing the action in term of classical and quantum angle fields $\theta_t = \frac{1}{2}[\theta_+(t) + \theta_-(t)]$ and $\hat{\theta}_t = \frac{1}{\hbar}[\theta_+(t) - \theta_-(t)]$. With some algebra the action is

$$\begin{aligned} S_K \;=\; & \frac{2i}{\hbar}\int_{t,t'} R^{-1}_{t,t'} \sin(\frac{\hbar}{2}\hat{\theta}_t)\cos(\frac{\hbar}{2}\hat{\theta}_{t'})\sin(\theta_{t'} - \theta_t) + \\ & \frac{4}{\hbar^2}\int_{t,t'} B_{t,t'} \sin(\frac{\hbar}{2}\hat{\theta}_t)\sin(\frac{\hbar}{2}\hat{\theta}_{t'})\cos(\theta_{t'} - \theta_t) \end{aligned} \tag{2.14}$$

2.3.1 Definition of Green's function and of renormalized dissipation

The renormalized retarded and correlation Green's functions are defined by

$$i\left\langle \hat{\theta}_{t'}\theta_t \right\rangle = R^R_{t,t'} \qquad \langle \theta_{t'}\theta_t\rangle = C^R_{t,t'} \qquad \langle\theta_t\rangle = v^R t. \tag{2.15}$$

Causality always ensures that always $\left\langle \hat{\theta}_t \right\rangle = \left\langle \hat{\theta}_t\hat{\theta}_{t'} \right\rangle = 0$. In the following we calculate (2.15) perturbatively in $1/\eta$. We identify the normalized dissipation by

$$\frac{1}{\eta^R} = \lim_{\omega\to 0}(-i\omega)R^R_\omega. \tag{2.16}$$

Which is equivalent to the definition by derivation of the velocity with respect to the external field,

$$\begin{aligned} \frac{1}{\eta^R} \;=\; & \frac{dv^R}{dE} = \frac{d}{dE}\left\langle \dot{\theta}_t \right\rangle = i\left\langle \int_{t'} \dot{\theta}_t\hat{\theta}_{t'} \right\rangle = \int_{t'} \frac{d}{dt}R^R_{t,t'} = \iint_{t',\omega} (-i\omega)R^R_\omega \mathrm{e}^{i\omega(t-t')} = \\ & \lim_{\omega\to 0} i\omega R^R_\omega = \lim_{\tau\to\infty} R_\tau \end{aligned} \tag{2.17}$$

2.4 Semi-classical limit of the action

For the semi-classical limit of the action, which is equivalent to the large η regime the quantum field is taken to the linear order $\sin(\frac{\hbar}{2}\hat{\theta}_t) \to \frac{\hbar}{2}\hat{\theta}_t$ and $\cos(\frac{\hbar}{2}\hat{\theta}_t) \to 1$. In this limit the retarded part of the action turn Gaussian while the correlation part remains a non-linear one. Therefore we solve this action perturbatively around S_0 which contain the retarded function.The model partition function is $Z = \int_{\mathcal{D}[\theta]} \mathrm{e}^{-S_0 - S_{int}}$ with

$$S_0 = i\int_{t,t'} \hat{\theta}_t R^{-1}_{t,t'}\theta_{t'} - iE\int_{t'}\hat{\theta}_{t'} = i\int_{\omega} R^{-1}_{\omega}\hat{\theta}_{\omega}\theta_{-\omega} - iE\int_{t'}\hat{\theta}_{t'} = i\int_{t,t'}\hat{\theta}_t R^{-1}_{t,t'}\delta\theta_{t'} \quad (2.18)$$
$$S_{int} = \frac{1}{2}\int_{t,t'} \hat{\theta}_t B_{t,t'}\hat{\theta}_t \cos(\theta_t - \theta_{t'}).$$

where in the last equality of S_0 we define $\theta_t = \delta\theta_t + vt$ with $v \equiv E/\eta$. With the above action the bare Green's functions are

$$i\left\langle \hat{\theta}_{t'}\theta_t \right\rangle_{S_0} = R_{t,t'} \qquad \langle\theta_t\rangle_{S_0} = vt = \frac{E}{\eta}t \qquad \langle\theta_{t'}\theta_t\rangle_{S_0} = 0 \quad (2.19)$$

In the following we calculate the renormalized Green's function up to second order in S_{int} which is equivalent to orders in $1/\eta$

$$R^R_{t,t'} = R_{t,t'} + R^{(1)}_{t,t'} + R^{(2)}_{t,t'} = R_{t,t'} + i\left\langle \hat{\theta}_{t'}\theta_t(-S_{int} + \tfrac{1}{2}S^2_{int})\right\rangle_{S_0}$$
$$C_{t,t'} = C^{(1)}_{t,t'} + C^{(2)}_{t,t'} = \left\langle \theta_{t'}\theta_t(-S_{int} + \tfrac{1}{2}S^2_{int})\right\rangle_{S_0} \quad (2.20)$$

and use it to identify the dissipation parameter.

2.4.1 Perturbation for the retarded function

The Retarded Green's function for first order in the perturbation

$$R^{(1)}_{t,t'} = i\left\langle \hat{\theta}_{t'}\theta_t(-S_{int})\right\rangle_{S_0} = -i\frac{1}{2}\int_{t_1,t_2} B_{t_1,t_2}\left\langle \hat{\theta}_{t_1}\hat{\theta}_{t_2}\cos(\theta_{t_1} - \theta_{t_2})\hat{\theta}_{t'}\theta_t\right\rangle_{S_0} \quad (2.21)$$

After derivation (appendix A.1) the function in frequency space is

$$R^{(1)}_{\omega} = R^2_{\omega}\int_{\omega_1} R_{\omega_1}\left[B^v_{\omega_1} - B^v_{\omega-\omega_1}\right] = R^2_{\omega}\int_t R_t B_t \cos vt\ (\mathrm{e}^{i\omega t} - 1) \quad (2.22)$$

where $B^v_\omega = \frac{1}{2}\left(B_{\omega+v} + B_{\omega-v}\right)$. and the renormalized η up to first order is

$$\begin{aligned}\frac{1}{\eta^R_1} &= \lim_{\omega\to 0}(-i\omega)R^{(1)}_\omega = \lim_{\omega\to 0}\frac{-i\omega}{(-i\omega)^2\eta^2}\int_t R_t B_t \cos vt \ (i\omega t) = \frac{\hbar}{2\pi\eta^2}\log(1+\omega_c^2/v^2) \\ &= -\frac{\hbar \log v/\omega_c}{\pi\eta^2} + \mathcal{O}(v) \end{aligned} \tag{2.23}$$

where $\omega_c = \eta/m$ is the high frequency cutoff. Using the same procedure $R^{(2)}_{t,t'} = \frac{i}{2}\left\langle \hat\theta_{t'}\theta_t(S_{int})^2\right\rangle$ (appendix A.2) we get

$$\begin{aligned} R^{(2)}_\omega = R^2_\omega \Bigg(&-\frac{1}{2}\int_t R_t B_t \cos vt \ (\mathrm{e}^{i\omega t}-1)\ \tilde C_t + \int_t R^{(1)}_t B_t \cos vt \ (\mathrm{e}^{i\omega t}-1) + \\ R_\omega &\left[\int_t R_t B_t \cos vt \ (\mathrm{e}^{i\omega t}-1)\right]^2 - \int_{t_1,t_2} R_{t_1}B_{t_1}B_{t_2}\sin vt_1 \sin vt_2 (1-\mathrm{e}^{i\omega t_1})t_1 \Bigg) \end{aligned} \tag{2.24}$$

with $\tilde C^{(1)}_\tau = 2(C^{(1)}_{t=0} - C^{(1)}_t)$. The renormalized η to second order

$$\frac{1}{\eta^R_2} = \frac{1}{\eta} - \frac{\hbar}{\pi\eta^2}\log v/\omega_c + \frac{\hbar^2}{\pi^2\eta^3}(\log^2 v/\omega_c + b_0 \log v/\omega_c) \tag{2.25}$$

where b_0 depends on the order of limits taken in the last expression, as explained in the following section.

2.4.2 Equilibrium and nonequilibrium limits

Taking into consideration the first three terms in the above expression for $R^{(2)}_\omega$ we find

$$\frac{1}{\eta^R_2} = \frac{1}{\eta} - \frac{\hbar}{\pi\eta^2}\log v/\omega_c + \frac{\hbar^2}{\pi^2\eta^3}(\log^2 v/\omega_c + b \log v/\omega_c) \tag{2.26}$$

The contribution of the forth term is unique since it depends on the order of limits taken. In a 'nonequilibrium' we need to derive the dissipation term in presence of the external force E. As defined in 2.16 we first take $\omega \to 0$ and then treat the field E as our RG cutoff, taking into consideration its logarithmic diverging contribution. The expression is then

$$\frac{1}{\eta^2}\lim_{v\to 0}\lim_{\omega\to 0}\frac{1}{i\omega}\int_{t1,t2} R_{t_1}B_{t_1}B_{t_2}\sin vt_1 \sin vt_2(1-\mathrm{e}^{i\omega t_1})t_1 =$$
$$-\frac{1}{\eta^3}\lim_{v\to 0}\int_{t1} R_{t_1}B_{t_1}\sin vt_1\ t_1^2\int_{t2} R_{t_2}B_{t_2}\sin vt_2 = \lim_{v\to 0}\frac{\hbar^2}{\pi^2\eta^3}\int^{\infty}\sin(vt_1)\times\int^{\infty}\sin(vt_2)/t_2^2 =$$
$$\lim_{v\to 0}\frac{\hbar^2}{\pi^2\eta^3}\frac{1}{v}\times v\log v+\mathcal{O}(v) = \frac{\hbar^2}{\pi^2\eta^3}\log v \tag{2.27}$$

With this contribution we find $b=0$. If instead we consider an equilibrium order of limits we take

$$\lim_{\omega\to 0}\lim_{v\to 0}\sin(vt_1)\sin(vt_2)=0 \tag{2.28}$$

and 2.27 vanishes. In this case $b=-1$ which recovers the known equilibrium results [17].

2.4.3 Perturbation in correlation function

The perturbative correlation function is similarly defined $C^{(1)}_\tau = \langle\theta_{t'}\theta_t(-S_{int})\rangle_{S_0}$

$$C^{(1)}_{t,t'} = -\frac{1}{2}\int_{t_1,t_2} B_{t_1,t_2}\left\langle\hat{\theta}_{t_1}\hat{\theta}_{t_2}\cos(\theta_{t_1}-\theta_{t_2})\theta_{t'}\theta_t\right\rangle =$$
$$\int_{t_1,t_2} B_{t_1,t_2}\cos v(t_1-t_2)R_{t,t_1}R_{t',t_2} \tag{2.29}$$

In Fourier space

$$C^{(1)}_\omega = |R_\omega|^2 B^v_\omega \tag{2.30}$$

because $C^{(1)}_{\tau=0}$ diverges it is useful to evaluate $\tilde{C}_{t,t'} = \langle[\theta_t-\theta_{t'}]^2\rangle$ which to 1st order is, with $\tau = t-t'$ ($\tau \ll 1/\omega_c$)

$$\tilde{C}^{(1)}_\tau = \int_\omega B^v_\omega |R_\omega|^2(1-\cos\omega\tau) \approx \frac{2\hbar}{\pi\eta}\begin{cases}\log(\eta\tau/m) & \tau<1/v \\ \\ \pi v\tau/2 & 1/v<\tau\end{cases}. \tag{2.31}$$

We can confirm that for $v=0$ in that order of η FDT is valid as

$$C^{(1)}_\omega|_{v=0} = \mathrm{Im}R(\omega)\hbar\ \mathrm{sign}(\omega) \tag{2.32}$$

2.5 Langevin Equation

The semiclassical action corresponds to the Langevin equation

$$m\ddot{\theta}_t + \eta\dot{\theta}_t = \xi^x_t \cos\theta_t + \xi^y_t \sin\theta_t + E \tag{2.33}$$

$$\left\langle \xi^i_\omega \xi^j_{\omega'} \right\rangle = B_\omega \delta(\omega+\omega')\delta_{ij} \qquad\qquad \left\langle \xi^i_t \right\rangle = 0$$

Where the noise term ξ^i_ω has a Gaussian weight. The MSR [35] method relates this Langevin equation and the semi-classical action Eq. (2.18). The partition function describing the Langevin equation is

$$Z = \int \mathcal{D}[\theta,\xi]\ \delta\left(m\ddot{\theta}_t + \eta\dot{\theta}_t - \xi^x_t \cos\theta_t - \xi^y \sin\theta_t - E\right) \mathrm{e}^{-|\xi_\omega|^2/2B(\omega)} \tag{2.34}$$

Introducing the 'quantum' field $\hat{\theta}$ by $\delta(X) = \int \mathcal{D}[\hat{\theta}]\mathrm{e}^{i\hat{\theta}_t X_t}$, and averaging over the noise field ξ^i will result in the semi classical partition function $Z = \int \mathcal{D}[\theta,\hat{\theta}]\mathrm{e}^{-S[\theta,\hat{\theta}]}$ where $S[\theta,\hat{\theta}] = S_0 + S_{int}$ is given by Eq. (2.18).

2.5.1 Numerical solution of the Langevin Equation

We solve the above Langevin equation numerically. The time is discretize to $t = T/N \times (1, 2, ...N)$, with T the total time span of system. The noise term ξ^i_t is generated numerically using a discrete Fourier transform of $\xi^i_\omega = \sqrt{B_\omega T}\mathcal{R}^i$ where $\mathcal{R}^i$ is a unit white Gaussian noise. The correlation function linearity requires introducing a high frequency cutoff τ_0. We choose the cutoff to be in Lorenzian form $B_\omega = \hbar\eta|\omega|/[1+\omega^2\tau_0^2]$, in the following section we explain the importance of this choice.

We solve the equation in iterative procedure. Using the convolution form

$$\theta_t = \int_{t'} R_{t,t'}\left[\xi^x_{t'} \cos\theta_{t'} - \xi^y_{t'} \sin\theta_{t'} - E\right] \tag{2.35}$$

starting with an arbitrary configuration of $\theta^{(0)}_t$ we calculate the equation RHS to find a new $\theta^{(1)}_t$. We repeat the procedure n times until the expression is saturated when $\theta^{(n)}_t = \theta^{(n+1)}_t$.

This procedure is improved if instead of taking the convolution result as the next order θ_t we use some mixing of that result and of the previous θ_t configuration in the form $\theta_t^{(m)} = (1-\beta)\theta_t^{(m-1)} + \beta \times \text{RHS}$ where β is mixing parameter. Typically n would be in order of 10^5 and $\beta = 0.1$.

2.5.2 Fluctuation dissipation relation and cutoff

The fluctuation dissipation relation requires $\mathrm{Im}R_\omega = \hbar \mathrm{sign}(\omega) B_\omega$. Adding cutoff τ_0 to the noise term B_ω requires adding it also to R_ω. Following [36] we choose

$$R_\omega^{-1} = -m\omega^2 - \frac{i\omega\eta}{1 - i\omega\tau_0} = -m\omega^2 + \delta R_\omega^{-1} \tag{2.36}$$

This choice of cutoff has the property that both R_ω and its inverse have no poles in the upper half plane. This property is required to ensure there is no causality breaking, the interaction cannot influence either the particle's past nor the environment's past. In the time domain this variant of the retarded functions rapidly oscillates for $\tau_0 \to 0$.

$$R_\tau = \Theta(\tau)\frac{1}{\eta}\left\{1 - \left[\frac{1-x^2}{2x}\sin(xt/2\tau_0) + \cos(xt/2\tau_0)\right]\mathrm{e}^{-t/2\tau_0}\right\} \quad x = \sqrt{\frac{4\eta\tau_0}{m} - 1}. \tag{2.37}$$

Fluctuation dissipation is satisfied for the correlation function

$$B_\omega = \frac{\hbar\eta\,|\omega|}{1 + \omega^2\tau_0^2}. \tag{2.38}$$

With this choice the Langevin equation takes the following form

$$\begin{aligned} m\ddot{\theta}_t &= \xi_t^x \cos\theta_t + \xi_t^y \sin\theta_t + E + \Delta_t \\ \Delta_t &= \frac{\eta}{\tau_0^2}\int_{-\infty}^{t} \sin[\theta_t - \theta_{t'}]\mathrm{e}^{-(t-t')/\tau_0} dt', \end{aligned} \tag{2.39}$$

where Δ_t is a correction term define by the term δR_ω^{-1} in the response function of Eq. (2.36) as $\int_{t'} \delta R_{t,t'}^{-1}[\xi_{t'}^x \cos\theta_{t'} + \xi_{t'}^y \sin\theta_{t'} + E] = -\int_\omega m\omega^2 \Delta_\omega$

In the numerical system we now have four time scales, the two numerical time scales, $\Delta\tau = T/N$ the time segment and T the time span, and the two physical high frequency cutoffs, τ_0 cutoff for the noise and ω_c mass cutoff. The region of interest where the velocity $v^R = \left\langle \dot{\theta}_t \right\rangle$ is between those time scales $\Delta\tau \ll \tau_0 < 1/\omega_c \ll 1/v^R \sim 1/v < T$. The inequality $\tau_0 < 1/\omega_c$ is useful since we compare the numerical result to an asymptotic result in which ω_c rather than $1/\tau_0$ is the high frequency cutoff.

2.5.3 Dissipation parameter

With the result for θ_t we can find the renormalized $1/\eta^R = dv^R/dE$ with $v^R = \left\langle \dot{\theta}_t \right\rangle$ where the average $\langle \cdot \rangle$ reflects an average on both the time domain $t > 1/\omega_c$ and on numerous realizations of the noise.

In the left panel of Fig.2 our numerical solution for the Langevin equation is shown, with a fit to the second order from both the nonequilibrium result $b_0 = 0$ and the equilibrium $b_0 = -1$. It is shown that the first is a better fit for the Langevin equation. When $1/v$ approaches the simulation time span T the numerics become unreliable, as T becomes the effective low frequency cutoff instead of the external field, and a plateau is observed at low E. In the right panel of Fig.2 we plot the the same data after subtraction of first order asymptotic results of Eq. (2.24).

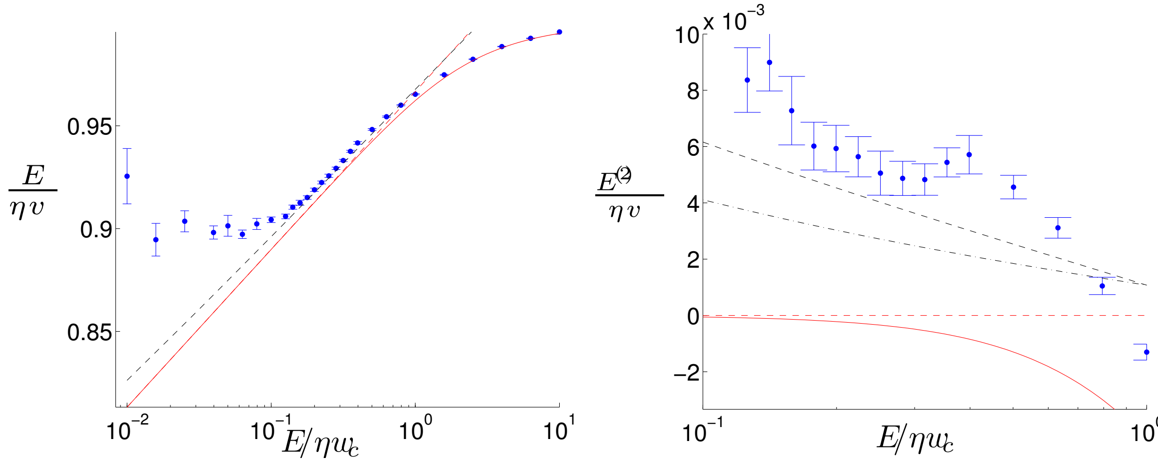

Figure 2: **Left panel**: Velocity-field relation for Eq. (2.39) with $\eta = 30\hbar/\pi$, $\omega_c = 100/\tau_0$ and $\tau_0 = 20\Delta\tau$. Here $N = 2^{15}$, $\Delta\tau = 1/20$. The circles are numerical data, the full red line is a 1st order perturbation in $1/\eta$, the dashed lower red line is its logarithmic expansion for large $\ln v/\omega_c$ and the dashed upper (black) line includes the 2nd order logarithmic term, corresponding to Eq. (2.25) for $b_0 = 0$.
Right panel: The same data after subtracting the 1st order terms, i.e. $\frac{E^{(2)}}{\eta v} = \frac{E}{\eta v} - 1 - \frac{\hbar}{\pi\eta}(\ln\frac{v}{\omega_c} - 1)$. An additional dash-dotted line corresponds to $b_0 = -1$, which is a worse fit to the data then $b_0 = 0$ (dashed upper line). Note that the numerical data displays E/v rather then dE/dv, hence Eq. (2.23) acquires a -1 term.

2.5.4 Fluctuation

With the numerical results for θ_τ we can create the correlation function $\tilde{C}_\tau^{(1)} = \langle[\theta_\tau - \theta_0]^2\rangle$, the first order perturbation for this correlation function is given in Eq. (2.31). In Fig.3 we plot the correlation function as a function of the time separation τ for the same parameters as in Fig.2, with and without a finite field. It is shown that for zero field the correlation has a subdiffusion logarithmic behavior while for finite force the correlation has a diffusion ($\sim \tau$) behavior.

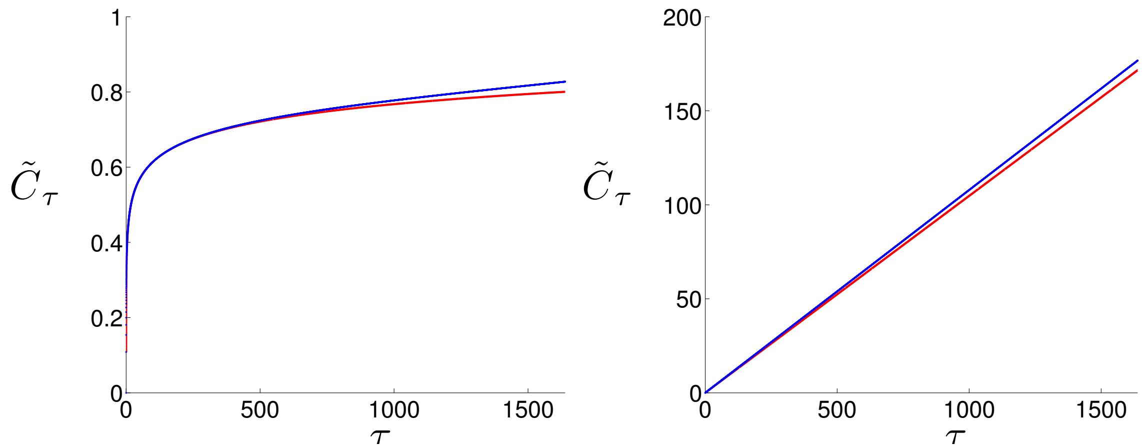

Figure 3: **Left panel**: The correlation function Eq. (2.32). and the asymptotic results of Eq. (2.32) (red) for $E = 0$. **Right panel**: The correlation function as a function of time (Blue) and the asymptotic results of Eq. (2.31) for $E/\eta = 1$ and $\tau_0 = 1$.

2.6 Dirty metal environment

A useful model for a noise producing environment is the CL [13] framework in which the environment is a set of oscillators, all linearly coupled to the particle. This is the environment we have used in the previous section. A more general way to characterize the environment is by its dielectric function, which for a dirty metal (DM) at low wavevector $\mathbf{q}$ and low frequency ω, is

$$\frac{1}{\varepsilon(\mathbf{q},\omega)} \approx \frac{-i\omega + D\mathbf{q}^2}{4\pi\sigma} \tag{2.40}$$

where D, σ are the diffusion coefficient and the conductivity, respectively.

In this section an equation of motion for a particle on the ring where the environment is that of a dirty metal is derived. The particle at position $\mathbf{r}(t)$ on the ring has a charge density $\rho(\mathbf{r},t) = e\delta^3(\mathbf{r} - \mathbf{r}(t))$, the energy in the system is $U = \int_{\mathbf{r}} \phi(\mathbf{r},t)\rho(\mathbf{r},t)$, where $\phi(\mathbf{r},t)$ is the potential. This potential is produced by the polarization of the environment by the particle, in momentum and frequency variables

$$\phi(\mathbf{q},\omega) = \alpha(\mathbf{q},\omega)\rho(\mathbf{q},\omega) \tag{2.41}$$

where the response function is defined in terms of dielectric function $\varepsilon(\mathbf{q},\omega)$

$$\alpha(\mathbf{q},\omega) = \frac{4\pi}{q^2\varepsilon(\mathbf{q},\omega)}. \tag{2.42}$$

We assume $\varepsilon(\mathbf{q},\omega)$ of the form (2.40). The retarded potential at the particle position $\mathbf{r}(t)$ is given as

$$\phi(\mathbf{r}(t),t) = \int_{t'}\int_{\mathbf{r}'}\alpha(\mathbf{r}(t)-\mathbf{r}',t-t')\rho(\mathbf{r}',t') = e\int_{t'}\alpha(\mathbf{r}(t)-\mathbf{r}(t'),t-t') \tag{2.43}$$

The force is $\mathbf{F} = -e\nabla\phi$, its projection on the azimuthal direction is $F_{\parallel}(t,\theta(t)) = -\frac{e}{R}\partial_\theta\phi(t,\theta_t)$, hence the equation of motion for the particle is

$$mR\ddot{\theta}(t) = -\frac{e}{R}\partial_{\theta_t}\phi(\theta_t,t) + f(\theta_t,t) \tag{2.44}$$

where $f(\theta_t,t)$ is a fluctuating force with zero average. The response in the position and time domains is

$$\alpha(\mathbf{X}=\mathbf{r}(t)-\mathbf{r}(t'),\tau=t-t') = \frac{1}{\sigma}\int_{\mathbf{q}}^{1/l}\int_\omega\left(\frac{-i\omega}{q^2}+D\right)\mathrm{e}^{-i\mathbf{q}\cdot\mathbf{X}-i\omega\tau} \tag{2.45}$$

The diffusion term yields $\delta(\tau)$, hence it is $\mathbf{X}$ independent and does not contribute to the force. With an elastic mean free path as a cutoff on the momentum $q \lesssim 1/l$ the first term is

$$\alpha(\mathbf{X}(\tau),\tau) = \frac{1}{4\pi\sigma}\frac{1}{\sqrt{\mathbf{X}^2+l^2}}\int_\omega(-i\omega)\mathrm{e}^{-i\omega\tau}$$

The displacement vector on a ring is $|\mathbf{X}| = \left|2R\sin\left(\frac{\theta_t-\theta_{t'}}{2}\right)\right|$. With the Fourier expansion $\left(4r^2\sin^2(z/2)+1\right)^{-1/2} = 1-\sum_{n=1}^{\infty}\alpha_n\sin^2(nz/2)$ where $r=R/l$, the azimuthal drag force is

$$\begin{aligned} F_{\parallel}(\theta_t,t) &= -\frac{e}{R}\partial_{\theta_t}\phi(\theta_t,t) = -\frac{e^2}{R}\partial_{\theta_t}\int_{t'}\alpha(\theta_t-\theta_{t'},t-t') = \\ & \frac{e^2}{4\pi\sigma}\frac{1}{Rl}\int_{t'}\left[\sum_n\frac{n}{2}\alpha_n\sin\left(n(\theta_t-\theta_{t'})\right)\right]\int_\omega(-i\omega)\mathrm{e}^{-i\omega(t-t')} \end{aligned} \tag{2.46}$$

The frequency integral equals $-\partial_{t'}\delta(t-t')$, integration by part of the last equation and the relation $\sum_n n^2\alpha_n = 2r^2$ yields

$$F_{\|}(\theta_t, t) = -\frac{e^2}{8\pi\sigma}\frac{1}{Rl}\sum_n n^2\alpha_n\dot{\theta}_t = -\frac{e^2}{4\pi\sigma}\frac{r^2}{Rl}\dot{\theta}_t \equiv -\eta R\dot{\theta}_t \tag{2.47}$$

This identifies the friction coefficient $\eta = \frac{e^2}{4\pi\sigma}\frac{1}{l^3}$. The Fluctuation dissipation theorem at zero temperature determines the symmetrized correlation function of ϕ

$$K_\phi(\mathbf{q},\omega) = -\hbar\,\mathrm{sign}(\omega)\mathrm{Im}\alpha(\mathbf{q},\omega) \tag{2.48}$$

$$K_\phi(\theta_t-\theta_{t'}, t-t') = \frac{\hbar}{4\pi\sigma l}\left[1-\sum_n \alpha_n \sin^2\left(\frac{\theta_t-\theta_{t'}}{2}n\right)\right]\int_\omega |\omega|\,\mathrm{e}^{-i\omega(t-t')}$$

The S_{int} term in the action 2.18 is the correlation function of the force, is a double differentiation on K_ϕ

$$\begin{aligned} S_{int}(\theta_t-\theta_{t'}, t-t') &= \frac{e^2}{R^2}\partial_{\theta_t}\partial_{\theta_{t'}}K_\phi(\theta_t-\theta_{t'}, t-t') = \\ & \frac{\hbar\eta}{2r^2}\sum_n n^2\alpha_n\cos\left(n(\theta_t-\theta_{t'})\right)B_{t,t'} \end{aligned} \tag{2.49}$$

This correlation is satisfied if the noise terms are as in the following Langevin equation

$$mR\ddot{\theta}_t + \eta R\dot{\theta}_t = \sum_n \frac{\sqrt{\alpha_n}n}{\sqrt{2}r}\{\xi_t^{n,x}\cos n\theta_t + \xi_t^{n,y}\sin n\theta_t\} + E \tag{2.50}$$

$$\left\langle \xi_\omega^{n,i}\xi_{\omega'}^{m,i}\right\rangle = \hbar\eta\,|\omega|\,\delta(\omega-\omega')\delta_{nm} \qquad i = x,y$$

and an external field E is added. Note that in the limit $r \ll 1$ the CL equation (2.33), is reached with $\alpha_1 = 2r^2$ and $\alpha_{n>1} = 0$.

2.6.1 Normalized dissipation

The normalized dissipation parameter as in 2.21 is,

$$\begin{aligned} R^{(1)}_{t,t'} &= i\left\langle \hat{\theta}_{t'}\theta_t(-S_{int})\right\rangle_{S_0} = \frac{1}{2i}\sum_n \frac{\alpha_n n^2}{2r^2}\int_{t_1,t_2} B_{t_1,t_2}\left\langle \hat{\theta}_{t_1}\hat{\theta}_{t_2}\cos n(\theta_{t_1}-\theta_{t_2})\hat{\theta}_{t'}\theta_t\right\rangle_{S_0} \\ &= \sum_n \frac{\alpha_n n^4}{2r^2}R_\omega^2\int_t R_t B_t\cos nvt\ (\mathrm{e}^{i\omega t}-1) \end{aligned} \tag{2.51}$$

the time integral gives $\sim \log nv/\omega_c = \log v/\omega_c + \text{const}$ and the dissipation parameter in first order is

$$\frac{1}{\eta_1^R} = -\frac{\hbar}{\pi\eta^2}\log v/\omega_c \sum_n \frac{\alpha_n n^4}{2r^2} + \mathcal{O}\big(v^0\big) = -\frac{\hbar}{\pi\eta^2}(1+9r^2)\log v/\omega_c + \mathcal{O}\big(v^0\big) \tag{2.52}$$

where we have used $\sum_n \frac{\alpha_n n^4}{2r^2} = \partial_z^4 \left(4r^2\sin^2(z/2)+1\right)^{-1/2}|_{z=0}$. Similar result for the dissipation in a dirty metal environment were derive by variational method [11].

2.6.2 Fluctuations

A first order approximation for the correlation functions at $E=0$ shows that the results are equal to those of approximation of Eq.(2.31) with a different prefactor, so that for $\tau \gg 1$

$$\tilde{C}_\tau = \frac{2\hbar}{\pi\eta r^2}\log\omega_c\tau \qquad \tau \gg 1/\omega_c \tag{2.53}$$

The numerical solution of this equation is done with a procedure similar to that in section 2.5. In this case a set of noise terms are created for each realization in the sum over n with the following results

2.7 Perturbation for the quantum action

In order to have a perturbation expansion in the full action of Eq. (2.14) one needs to identify a Gaussian term S_0 within the action, then a perturbation can be done for around S_0. Similar to the classical case perturbation in the noise term is of interest, here we also need to consider the non-Gaussian part of the retarded term in the action. A simplification of the action can be done by the use of the retarded function expression

$$\begin{aligned} R^{-1}_{t,t'} &= \delta(t-t')[m\partial_t\partial_{t'} + \eta\partial_{t'}] \\ R^{-1}_{t,t'}\sin(\frac{\hbar}{2}\hat{\theta}_t)\cos(\frac{\hbar}{2}\hat{\theta}_{t'})\sin(\theta_{t'}-\theta_t) &= \delta(t-t')[m\dot{\hat{\theta}}_t\dot{\theta}_t + \frac{\eta}{\hbar}\sin(\hbar\hat{\theta}_t)\dot{\theta}_t] \end{aligned} \tag{2.54}$$

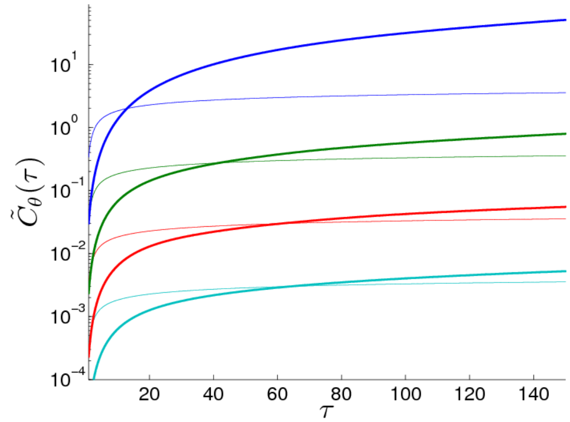

Figure 4: The correlation functions as a function of time (bold) and the perturbation (2.53) (thin) for $r = 10^x$, $x = -1, -0.5, 0, 0.5, 1$, $E = 0$, $\hbar/(\pi\eta) = 10^{-2}$. The curves are from the highest to the lowest for increasingly larger r.

with that the action becomes

$$S_K = i \int_t [m\dot{\hat{\theta}}_t \dot{\theta}_t + \frac{\eta}{\hbar}\sin(\hbar\hat{\theta}_t)\dot{\theta}_t] + \frac{2}{\hbar^2}\int_{t,t'} B_{t,t'} \sin(\frac{\hbar}{2}\hat{\theta}_t)\sin(\frac{\hbar}{2}\hat{\theta}_{t'})\cos(\theta_{t'} - \theta_t) \tag{2.55}$$

It is useful to use the two-cutoff response as in Eq. (2.36) with $R_\omega^{-1} = -m\omega^2 + \delta R_\omega^{-1}$, where $\delta R_\omega^{-1} = \frac{-i\omega\eta}{1-i\omega\tau_0}$, hence

$$\begin{aligned} \delta R_{t,t'}^{-1} &= \partial_{t'} \int_\omega \frac{-\eta}{1 - i\omega\tau_0} \mathrm{e}^{-i\omega(t-t')} = -\frac{\eta}{\tau_0}\partial_{t'}[\mathrm{e}^{-(t-t')/\tau_0}\Theta(t-t')] = \\ &\frac{\eta}{\tau_0}\mathrm{e}^{-(t-t')/\tau_0}\Theta(t-t')\partial_{t'} \end{aligned} \tag{2.56}$$

for $\tau_0 \to 0$ we have $\delta R^{-1}_{t,t'} \to \eta\delta(t-t')\partial_{t'}$. This operator identity is satisfied for any function decaying faster then $\mathrm{e}^{|t'|/\tau_0}$ at $t' \to -\infty$. Adding the external force E the action is then

$$
\begin{aligned}
S_K &= S_0 + S_{int} + S_c \\
S_0 &= i\int_{t,t'} \hat{\theta}_t R^{-1}_{tt'}\theta_{t'} - iE\int_t \hat{\theta}_t = i\int_{t,t'} \hat{\theta}_t R^{-1}_{tt'}\delta\theta_{t'} \\
S_{int} &= \frac{2}{\hbar^2}\int_{t,t'} B_{t,t'}\sin(\frac{\hbar}{2}\hat{\theta}_t)\sin(\frac{\hbar}{2}\hat{\theta}_{t'})\cos(\theta_{t'}-\theta_t) \\
S_c &= \frac{2i}{\hbar}\int_{t,t'} \delta R^{-1}_{t,t'}[\sin(\frac{\hbar}{2}\hat{\theta}_t)\cos(\frac{\hbar}{2}\hat{\theta}_{t'})\sin(\theta_{t'}-\theta_t) - \frac{\hbar}{2}\hat{\theta}_t\theta_{t'}]
\end{aligned}
\tag{2.57}
$$

for $\tau_0 \to 0$ S_c reduces back to $\frac{i\eta}{\hbar}\int_t[\sin(\hbar\hat{\theta}_t)-\hbar\hat{\theta}_t]\dot{\theta}_{t^-}$ as in Eq. (2.55) where t^- in the last line is infinitesimal below t so that the retarded nature of $R^{-1}_{t,t'}$ is maintained. For example averaging over perturbation, which are the non-connected terms has to vanish $\langle S_c\rangle_{S_0} \sim R_{t=0^-} = 0$, this is a crucial point for the perturbation expansion in the following. The bare Green's function is now as in the classical perturbation of Eq. (2.19).

2.7.1 Perturbations: 1st order

The retarded Green's function for the first order in S_{int} follows a similar procedure to that of the classical case (appendix A.1)

$$
\begin{aligned}
R^{(1)}_{t,t'} &= \frac{-2i}{\hbar}\left\langle \hat{\theta}_{t'}\theta_t \int_{t_1,t_2} B_{t_1,t_2}\sin(\frac{\hbar}{2}\hat{\theta}_t)\sin(\frac{\hbar}{2}\hat{\theta}_{t'})\cos(\theta_{t_1}-\theta_{t_2})\right\rangle_{S_0} = \\
&\frac{-2i}{\hbar}\sum_{\sigma,\sigma',\mu=\pm}\frac{\sigma\sigma'}{2^4}\partial_{\alpha_1}\partial_{\alpha_2}|_0\int_{t_1,t_2} B_{t_1,t_2}\left\langle \mathrm{e}^{i\frac{\hbar}{2}(\alpha_1\hat{\theta}_{t'}+\alpha_2\theta_t+\sigma\hat{\theta}_{t_1}+\sigma'\hat{\theta}_{t_2})+i\mu(\theta_{t_1}-\theta_{t_2})}\right\rangle_{S_0} = \\
&= -\frac{2}{\hbar}\int_{t_1,t_2} B_{t_1,t_2}\sin(\frac{\hbar}{2}R_{t_1,t_2})[R_{t_1,t'}-R_{t_2,t'}]R_{t,t_1}\cos v(t_1-t_2)
\end{aligned}
\tag{2.58}
$$

In frequency space the retarded Green's function is the same as the classical with the replacement $\frac{\hbar}{2}R_t \to \sin(\frac{\hbar}{2}R_t)$

$$
R^{(1)}_\omega = R^2_\omega\frac{2}{\hbar}\int_t \sin(\frac{\hbar}{2}R_t)B_t\cos vt\ (\mathrm{e}^{i\omega t}-1)
\tag{2.59}
$$

and the renormalized dissipation coefficient η up to first order is

$$\begin{aligned}\frac{1}{\eta_1^R} &= \lim_{\omega\to 0}(-i\omega)R^{(1)}_\omega = \lim_{\omega\to 0}\frac{-i\omega}{(-i\omega)^2\eta^2}\frac{2}{\hbar}\int_t \sin(\frac{\hbar}{2}R_t)B_t\cos vt\ (i\omega t) = \\ & \frac{2}{\pi\eta}\sin(\frac{\hbar}{2\eta})\int_{\omega_c}^{v}\frac{dt}{t} + \mathcal{O}(v) = -\frac{2}{\pi\eta}\sin(\frac{\hbar}{2\eta})\log v/\omega_c + \mathcal{O}(v) \end{aligned} \tag{2.60}$$

so at the semi-classical limit which is equivalent to large η limit we retrieve the previous result (2.23). First order perturbation in S_c vanish

$$\left\langle \hat{\theta}_{t'}\theta_t S_c\right\rangle_{S_0} = \left\langle \hat{\theta}_{t'}\theta_t \frac{i\eta}{\hbar}\int_{t_1}[\sin(\hbar\hat{\theta}_{t_1}) - \hbar\hat{\theta}_{t_1}]\dot{\theta}_{t_1^-}\right\rangle_{S_0} = 0 \tag{2.61}$$

since the Green's function of the types $\langle\theta\theta\rangle_{S_0}$ and $\left\langle\hat{\theta}\hat{\theta}\right\rangle_{S_0}$ vanish, i.e. the only possible connected term is composed of multiplication of the Green's function of the type $\left\langle\hat{\theta}\theta\right\rangle$. Expanding the *sin* in the expression the first order vanishes with the second term and the expression is $\sim \left\langle \hat{\theta}_{t'}\theta_t\dot{\theta}_{t_1^-}[-\hbar^3\hat{\theta}^3_{t_1} + \frac{1}{5}\hbar^5\hat{\theta}^5_{t_1} + ...]\right\rangle_{S_0}$ separating the angular fields to pairs (Wick extraction) it is clear that there exists no term without $\left\langle\hat{\theta}\hat{\theta}\right\rangle_{S_0} = 0$. With the same reasoning $\left\langle\hat{\theta}_{t'}\theta_t S_c^2\right\rangle_{S_0} = 0$. The mixed term $\left\langle\hat{\theta}_{t'}\theta_t S_c S_{int}\right\rangle_{S_0}$ does not vanish, but as S_c contribute $\sim\hbar^3$ at least and $S_{int}\sim\hbar$ this term is of $\mathcal{O}(\hbar^3)$ where our perturbation will only be to $\mathcal{O}(\hbar^2)$.

2.7.2 Perturbations: 2nd order

In section (B.1) the second order perturbation $R^{(2)}_{t,t'} = \frac{i}{2}\left\langle\hat{\theta}_{t'}\theta_t S^2_{int}\right\rangle_{S_0}$ of the retarded Green's function and the dissipation parameter are derived,

$$\begin{aligned}\frac{1}{\eta_2^R} &= \frac{i}{2\hbar^4\eta^2}\frac{\partial}{\partial v}\sum_{\epsilon_i,\mu=\pm}\int_{t_2,t_3,t_4} B_{t_1,t_2}B_{t_3,t_4}\epsilon_2\epsilon_3\epsilon_4 A_2\sin[v(t_1-t_2)+\mu v(t_3-t_4)] \\ A_2 &= \exp\{i\frac{\hbar}{2}\epsilon_2(R_{t_1,t_2}+\mu R_{t_3,t_2}-\mu R_{t_4,t_2}) + i\frac{\hbar}{2}\epsilon_3(R_{t_1,t_3}-R_{t_2,t_3}-\mu R_{t_4,t_3})\}\times \\ & \exp\{i\frac{\hbar}{2}\epsilon_4(R_{t_1,t_4}-R_{t_2,t_4}+\mu R_{t_3,t_4})\}\end{aligned} \tag{2.62}$$

we could not compute this expression with the retarded function $R_\tau = \frac{1}{\eta}(1-\mathrm{e}^{-\eta\tau/m})\Theta(\tau)$. In general we have two high frequency cutoffs $m/\eta, \tau_0$ in Eq. (2.36) and we assume that the resulting small frequency expression is independent of their ratio. The strict limit $m \to 0$ leads to large oscillations at $t \to 0$ so we do not attempt this limit. We define a formal cutoff time $\tau_1(m/\eta, \tau_0)$ for R_τ and we first take the formal limit $\tau_1 \to 0$; we keep, however, the cutoff τ_0 in $B(\omega)$, Eq. (2.38). In this limit $R_\tau \to \frac{1}{\eta}\Theta(t)\mathrm{e}^{-\delta t}$ where $\delta \to 0^+$ to ensure the retarded nature (poles of $1/(\omega + i\delta)$). With that we find (section B.1)

$$\frac{1}{\eta^R} = \frac{1}{\eta^R} - \frac{2}{\pi\eta}\sin(\frac{\hbar}{2\eta})[\ln(v\tau_0)+1] + \frac{4}{\pi^2\hbar}\sin^2(\frac{\hbar}{2\eta})\sin(\frac{\hbar}{\eta})\cdot[\ln^2(v\tau_0)+3\ln(v\tau_0)] \quad (2.63)$$

Here $1/\eta_2^R$ is calculated in a formal limit $\tau_1 \to 0$. Note that the result is finite, and in fact even ∂_{τ_1} at $\tau_1 \to 0$ is finite. Instead of the calculation of (2.62) for $\tau_1 \neq 0$ we consider now the general structure for perturbations to 2nd order which has 3 terms: ln terms for both 1st and 2nd order terms, and $\ln^2$ only from the 2nd order,

$$\begin{aligned}\frac{1}{\eta^R} &= \frac{1}{\eta} + a(\eta,\tau_0 v,\tau_1 v)\ln[\bar{a}(\eta,\tau_0 v,\tau_1 v)\tau_0 v] + b(\eta,\tau_0 v,\tau_1 v)\ln^2[\bar{b}(\eta,\tau_0 v,\tau_1 v)\tau_0 v] \\ &\quad + c(\eta,\tau_0 v,\tau_1 v)\ln[\bar{c}(\eta,\tau_0 v,\tau_1 v)\tau_0 v] \end{aligned} \quad (2.64)$$

Note that $\tau_1 = \tau_1(m/\eta, \tau_0)$ and $\tau_1(m/\eta, \tau_0 = 0) = m/\eta$. We assume now cutoff universality, i.e. one can take the limit of both $\tau_0, m/\eta \to 0$ with any fixed ratio without affecting the result, however, the case $\tau_0 \neq 0$, $m/\eta \to 0$ is avoided as it leads to diverging oscillations in (2.37). Note that $\tau_1 = 0$, $\tau_0 \neq 0$ is not realized by any τ_0, m/η, yet, we use the mathematical fact that this limit is well defined. Hence we expand the $a, b, c, \bar{a}, \bar{b}, \bar{c}$ functions in τ_1. Since $\tau_0 \to 0$ is well defined, there are no terms $\sim \frac{\tau_1}{\tau_0}$ in a, b, c and no $\sim (\frac{\tau_1}{\tau_0})^2$ in $\bar{a}, \bar{b}, \bar{c}$. We use the known limit $\tau_1 = 0$ to identify the coefficients

$$\begin{aligned}\frac{1}{\eta^R} &= \frac{1}{\eta} + a(\eta)\{\ln[\tau_0 v + \bar{a}_2(\eta)\tau_1 v] + 1\} + \\ &\quad b(\eta)\{\ln^2[\tau_0 v + \bar{b}_2(\eta)\tau_1 v] + 3\ln[\tau_0 v + \bar{c}_2(\eta)\tau_1 v] + 1\} \\ a(\eta) &= \frac{-2}{\pi\eta}\sin(\frac{\hbar}{2\eta}) \\ b(\eta) &= \frac{4}{\pi^2\hbar}\sin^2(\frac{\hbar}{2\eta})\sin(\frac{\hbar}{\eta}) \end{aligned} \quad (2.65)$$

In short, this form reproduces the known $\tau_1 = 0$ limit, and also allows for a finite τ_1 consistent with the general form of $\tau_0 v, \tau_1 v \ll 1$. In particular there are no terms $\sim \ln \tau_1 v$ when $\tau_0 \neq 0$ since they diverge at $\tau_1 \to 0$. Now take the limit $\tau_0 = 0, \tau_1 = m/\eta$ of a single cutoff

$$\frac{1}{\eta^R} = \frac{1}{\eta} + a(\eta)\{\ln[\bar{a}_2(\eta)\frac{m}{\eta}v] + 1\} + b(\eta)\{\ln^2[\bar{b}_2(\eta)\frac{m}{\eta}v] + 3\ln[\bar{c}_2(\eta)\frac{m}{\eta}v] + 1\} \tag{2.66}$$

The $\bar{a}_2(\eta), \bar{b}_2(\eta), \bar{c}_2(\eta)$ are shifts of the classical cutoff, thus they should not depend on the quantum $\hbar/\eta$, hence these are constants. Another way to see this, is by $\eta^R(\frac{m}{\eta}v = 1) = \eta$ so that

$$\begin{aligned} &\ln \bar{a}_2(\eta) + 1 \lesssim 1 \\ &\ln^2 \bar{b}_2(\eta) + 3\ln \bar{c}_2(\eta) + 1 \lesssim 1 \end{aligned} \tag{2.67}$$

η^R is up to ln accuracy, so matching at $vm/\eta = 1$ can miss a $\mathcal{O}(1)$ terms. In fact $1st$ order terms has $\ln[\bar{a}_2(\eta)] = -1$. Therefore

$$\frac{1}{\eta^R} = \frac{1}{\eta} - \frac{2}{\pi\eta}\sin(\frac{\hbar}{2\eta})\ln[\frac{m}{\eta}v] + \frac{4}{\pi^2\hbar}\sin^2(\frac{\hbar}{2\eta})\sin(\frac{\hbar}{\eta})\{\ln^2[\frac{m}{\eta}v] + b_0\ln[\frac{m}{\eta}v]\} \tag{2.68}$$

with $b_0 \lesssim 1$.The main conclusion is that there is a new small parameter in the perturbation series, $\sin(\frac{\hbar}{2\eta})$. The perturbation is formally in $R^{2n-1}B^n/\eta^2 \sim \hbar^n/\eta^{n+1}$ for large η, but in the present scheme R^{2n-1} factors in front of the logarithmic term become periodic functions.

2.8 Renormalization Group treatment

We performed in section (2.7) a perturbative expansion of the action with respect to S_{int}, S_c to compute η^R. The small $1/\eta$ form of each term is $1/\eta^{n+1}$. The perturbative expansion of η^R exhibits logarithmic divergences when $E \to 0$, thus the velocity $v = E/\eta$ provides a natural low frequency cutoff for this divergences, and the mass provides a high frequency cutoff at $\omega_c = \eta/M$, or alternatively τ_0 provide this cutoff. The expansion terms can be classified as n-loops RG expansion if they satisfy the Lie's equation [37]

$$\frac{d}{d\ln v}g^R(v/\omega_c, g) = \frac{d}{d\ln \xi}g^R(\xi, g^R(v/\omega_c, g))|_{\xi=1} \tag{2.69}$$

The renormalization procedure consists of a rescaling of the frequency cutoff v. The high frequency cutoff ω_c is replaced by v as the frequencies in the range ω_c to v have been integrated out to produce the effective dissipation parameter $\eta^R(v)$. $\eta^R(v)$ is the only parameter that is renormalized in the procedure, its bare value corresponds to $\eta = \eta^R(\omega_c)$.

In the limit of large η which is the same as the semiclassical result of Eq. (2.25), we can express Eq. (2.68) in terms of the small parameter $g = \frac{\hbar}{\pi\eta}$ and $g_R = \frac{\hbar}{\pi\eta^R(E)}$ and obtain

$$g^R = g - g^2 \ln[v/\omega_c] + g^3 \left\{\ln^2[v/\omega_c] + b_0 \ln[v/\omega_c]\right\} \tag{2.70}$$

This satisfy Lie's equation. A direct way to see that is to define the β function in two ways, via a derivative at $\ln = 0$ with $g \to g^R$

$$\beta = \frac{dg^R}{-d\ln v} = (g^R)^2 - b_0(g^R)^3 = (g - g^2\ln(v/\omega_c))^2 - b_0 g^3 + O(g^4) \tag{2.71}$$

and the other way is a direct derivative of (2.70)

$$\beta = \frac{dg^R}{-d\ln v} = g^2 - g^3(2\ln v/\omega_c + b_0) \tag{2.72}$$

and check that the two results coincide.

For the quantum theory, beyond large η we find, due to the periodicity of the action in the angle variables, that the R^{2n-1} factors in front of the logarithmic terms have become a periodic functions. We note that in (2.68) $g = \frac{2}{\pi}\sin\frac{\hbar}{2\eta}$ acts as an unexpected small parameter for the expansion. Since all divergences vanish when $g = 0$ it raises the interesting possibility that $g = 0$ be viewed as a RG fixed point. For that we need to find a renormalized coupling which obeys multiplicative RG, the simplest choice being $g^R = \frac{2}{\pi}\sin\frac{\hbar}{2\eta^R}$. The question is then whether all ln terms of the β-function $\beta = -E\partial_E g^R$ can be written in terms of g^R. Although the non-periodic $1/\eta$ factor in (2.68) appears at first problematic, we propose that resummation from higher loops, which allows for higher order terms $\mathcal{O}\left(\frac{1}{\eta^4}\right)$ changes the 1-loop term in (2.68) by $\frac{1}{\eta} \to \sin\frac{1}{\eta}$, so that by taking a sine of both sides it yields to order

g^3

$$\sin\frac{\hbar}{2\eta^R} = \sin\frac{\hbar}{2\eta} - \frac{2}{\pi}\sin^2(\frac{\hbar}{2\eta})\cos(\frac{\hbar}{2\eta})\ln[\frac{v}{\omega_c}] + \frac{4}{\pi^2}\sin^3\frac{\hbar}{2\eta}\cos^2\frac{\hbar}{2\eta}\{\ln^2[\frac{v}{\omega_c}] + b_0\ln[\frac{v}{\omega_c}]\} \tag{2.73}$$

with the above definition for g, near $g=0$ fixed points

$$g^R = g \pm g^2\ln(v/\omega_c) + g^3[\ln^2(v/\omega_c) + b_0\ln(v/\omega_c)] \tag{2.74}$$

where $\pm$ refers to $g=0$ with $\cos\frac{1}{\eta} = \pm 1$. Therefore $\beta(g^R) = \mp(g^R)^2 - b_0(g^R)^3 + \mathcal{O}\big((g^R)^4\big)$.

2.8.1 Alternative response functions

To further motivate the last proposal we consider the response function $\bar{R}_{t,t'} = i\frac{2}{\hbar}\left\langle \theta_t \sin(\frac{\hbar}{2}\hat{\theta}_{t'})\right\rangle$. Physically, $\mathrm{e}^{\pm\frac{\hbar}{2}i\hat{\theta}_{t'}}$ corresponds to an electric field pulse $\delta E(t) = \pm\delta(t-t')$ or equivalently a rapid change of flux by $\pm\frac{1}{2}$, therefore $\bar{R}_{t,t'}$ corresponds to the difference in response to these two flux pulses. For $\bar{R}_{t,t'}$ the 1-loop term is fully periodic with $\frac{\hbar}{2\eta} \to \sin(\frac{\hbar}{2\eta})$ as in Eq. (2.68).

We note that there are many other operators that have vanishing perturbations at $g=0$ to 2nd order in S_{int}, S_c. E.g we can define an effective η^R using the dissipation term in Eq. (2.57) for which the relevant response function is $\left\langle \theta_t \sin(\hbar\hat{\theta}_{t'})\right\rangle$. Another option is to consider the response to an ac field that rotates in resonance with the particle, i.e. $E_{ac}(\sin vt, -\cos vt)$, in addition to the DC field E. The Hamiltonian is then

$$\mathcal{H}_{ac} = E_{ac}\{\sin vt\cos[vt+\delta\theta(t)] - \cos vt\sin[vt+\delta\theta(t)]\} = -E_{ac}\sin\delta\theta(t) \tag{2.75}$$

and the Keldysh action $S_{ac} = E_{ac}[\sin\delta\theta^+(t) - \sin\delta\theta^-(t)] = 2E_{ac}\cos\delta\theta_t\sin\frac{\hbar}{2}\hat{\theta}_t$ and the response function is

$$\frac{\partial v}{\partial E_{ac}} = \lim_{t-t'\to\infty}\bar{\bar{R}}_{t,t'} \qquad \Rightarrow \qquad \bar{\bar{R}}_{t,t'} = 2i\langle\theta_t\cos\delta\theta_{t'}\sin\frac{\hbar}{2}\hat{\theta}_{t'}\rangle \tag{2.76}$$

In both cases one can compute an RG equation as in Eq. (2.74). With the above reasoning we suggest that $g=0$ are exact zeroes of the perturbation expansion and requiring an RG structure leads then to the result (2.74).

Eq. (2.74) yields fixed points at $\frac{\hbar}{2\eta_n} = n\pi$ with $n = 1, 2, 3, ...$ that are attractive at $\eta > \eta_n$ and repulsive at $\eta < \eta_n$, i.e. the flow of $\eta \neq \eta_n$ is always to smaller η. However at these fixed points a Gaussian evaluation yields the correlation for large t

$$\langle \cos\theta_t \cos\theta_0 \rangle = \frac{1}{2} \mathrm{e}^{-\frac{1}{2}\tilde{C}_t} \sim \mathrm{e}^{-\frac{\hbar}{\pi\eta}\log\omega_c t} = t^{-2n}. \tag{2.77}$$

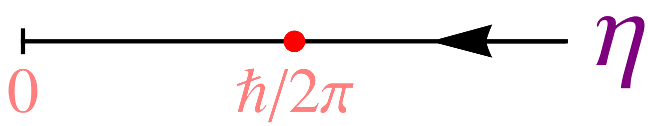

Figure 5: The flow diagram for η. For $\eta > \eta_c = \hbar/(2\pi)$ the dissipation parameter flows to η_c.

There is a theorem for the lattice model [38] where the equilibrium action with mass related cutoff is replaced by an action on a lattice resulting in an XY model with long range interactions. The theorem states [38] that $\langle \cos\theta_t \cos\theta_0 \rangle \sim 1/t^2$; this result was also derived in first order in η [10, 11]. The range $\eta > \eta_1$ has an RG flow to η_1 and is therefore consistent with the theorem. The hypothesis of Gaussian fixed points corresponding to $n \geq 2$ is inconsistent with the theorem, i.e. $\langle \cos\theta_t \cos\theta_0 \rangle$ becomes a relevant operator at the $n \leq 2$ points rendering them unstable. For $\eta < \eta_1$ the system may have non-gaussian fixed points or a line of fixed points as hinted by the small η perturbation [10, 11].

2.9 Conclusions

We conclude that for $\eta > \eta_1 \equiv \eta^R$ the SEB satisfies the quantization

$$\int_0^1 \frac{C_0^2(N_0)}{C_g^2} R_q(N_0) dN_0 = \frac{h}{e^2}\,. \tag{2.78}$$

In particular, when $\eta/\hbar \gtrsim 1$ we have from the known $M^*/M \sim \mathrm{e}^{\pi\eta/\hbar}$ [7–11] and from Eq. (2.8) that $C_0/C_g = 1 + O(\mathrm{e}^{-\pi\eta/\hbar})$. We expect R_q to be independent of N_0 at large η, hence

$$R_q = \frac{h}{e^2}[1 + O(\mathrm{e}^{-\pi\eta/\hbar})] \tag{2.79}$$

similar to the $N_c = 1$ case [5].

The conductance for the ring can be defined by the voltage around the ring $2\pi E/e$ and the current $e\langle\dot{\theta}\rangle/2\pi$, hence we expect the conductance for $\eta > \eta^R$ to be:

$$G_{ring} = \frac{e^2}{4\pi^2\eta^R} = \frac{e^2}{h}\,. \tag{2.80}$$

In section 5 we propose an experiment to verify this result.

2.9.1 Intuitive argument for the quantization

The special value $\eta^R = \hbar/(2\pi)$ has a topological interpretation as a Thouless charge pump [39]. Consider a slow change of ϕ_x by one unit with $\hbar\dot{\phi}_x = \eta^R\langle\dot{\theta}\rangle$. For this special value $\eta^R = \hbar/(2\pi)$ the total change in the position of the particle $\int_t\langle\dot{\theta}\rangle dt = 2\pi$, i.e. the particle comes back to the same position on the ring and a unit charge has been transported. Such quantization has been shown for cases where the spectrum has a gap [39], though quantized charge transport was shown also in cases without a gap [40, 41]. The quantized η^R also results from arguing that there should be a unique frequency $\omega_E = v$ as $E \to 0$, as suggested by linear response.

3 Equilibrium study of particle dissipation

In this section we study a particle on ring in equilibrium in the presence of a dissipative CL environment and derive the effective dissipation parameter η^R dependence on the bare η parameter, on the external flux ϕ_x and on the temperature.

Our aim is to calculate the particle equilibrium Green's function $K_\omega = \left\langle |\theta_\omega|^2 \right\rangle$. Following section 2.2 we define the low frequency expansion terms of K_ω

$$K_\omega = K_0(\phi_x) + |\omega| \, K_1(\phi_x) + \mathcal{O}\big(\omega^2\big) \tag{3.1}$$

The first order is the effective mass $K_0(\phi_x) = \hbar/M^*(\phi_x)$ and we have seen that by taking an average over the flux this term vanishes $\int_0^1 K_0(\phi_x) d\phi_x = 0$. The interest is in the second term $\hbar/\eta^R = \int_0^1 K_1(\phi_x) d\phi_x$, for which in section 2.8 we found its normalization dependence on the bare $1/\eta$ parameter. We also expect that the important contribution for the effective dissipation term is around the degeneracy point $\phi_x = 1/2$.

In this section we use the Equilibrium Matsubara formalism to calculate $K_\omega(\phi_x)$ with two types of perturbation schemes as well as by numerical MC methods, with the aim of identifying $K_1(\phi_x)$ in order to identify the quantization of the noise described in the previous section, Eq. (2.7). We expect that for sufficiently low temperatures the flux integrated linear response $\int_0^1 K_1(\phi_x)$ will be universal for any value of $\eta > \hbar/(2\pi)$.

In section 3.1 we define the action model for the system and the Green's function, in section 3.2 we calculate the Green's function perturbatively up to first order for both small and large dissipation parameter. In section 3.3 we extend the perturbative calculation for finite T and taking into consideration all windings.

We wish to solve this model numerically using the MC methods with the aim to identify numerically $K_1(\phi_x)$. In section 3.4 we describe the numerical MC method used to solve the model, and our chosen implementation of the numerics, and its limitation. In 3.5 we show and discuss the MC result and its limitation.

3.1 Model action

The time dependent angular position $\theta_\tau^{(m)}$ of a particle on a ring has in general a winding m to describe the number of times the particle encircles the ring so that $\theta_\tau^{(m)} = \theta_\tau + 2\pi m\tau T$, where $\theta_0 = \theta_{1/T}$ has a periodic boundary condition. In presence of external flux ϕ_x (in units of flux quantum hc/e) the partition sum has the form

$$Z = \sum_m \mathrm{e}^{2\pi i m\phi_x} \int \mathcal{D}[\theta] \mathrm{e}^{-S^{(m)}[\theta]} \tag{3.2}$$

With the action describing a CL environment

$$\begin{aligned} S^{(m)}[\theta] &= \frac{1}{2} M \int_0^{1/T} (\dot{\theta}_\tau^{(m)})^2 d\tau - 2\int_0^{1/T}\int_0^{1/T} g_{\tau,\tau'} \sin^2\left(\frac{\theta_\tau^{(m)} - \theta_{\tau'}^{(m)}}{2}\right) \\ g_\tau &= -\frac{\alpha}{2}\frac{(\pi T)^2}{\sin^2 \pi T\tau} \qquad \tau \neq 0 \end{aligned} \tag{3.3}$$

where $\alpha = \eta/\pi$ with our previous η (α notation is conventional here).

3.2 Perturbation scheme

Here we will consider a general perturbation scheme for any partition function $Z = \int_{\mathcal{D}[\theta]} \mathrm{e}^{-S}$, with action $S = S_0 + S_{int}$ where

$$S_0 = \frac{1}{2\beta} \sum_\omega G_\omega^{-1} |\theta_\omega|^2 \tag{3.4}$$

is a Gaussian action and $G_\omega = \omega^2 \tilde{K}_\omega$ is zero order Matsubara Green's function. Expanding in S_{int}, the Green's function up to first order in the interaction parameter is

$$\begin{aligned} G_\omega^{(1)} = \frac{\int_{\mathcal{D}[\theta]} |\theta_\omega|^2 \mathrm{e}^{-S_0}(1 - S_{int})}{\int_{\mathcal{D}[\theta]} \mathrm{e}^{-S_0}(1 - S_{int})} = \\ G_\omega - \frac{\int_{\mathcal{D}[\theta]} \mathrm{e}^{-S_0} \left(|\theta_\omega|^2 - G_\omega\right) S_{int}}{\int_{\mathcal{D}[\theta]} \mathrm{e}^{-S_0}} + \mathcal{O}(S_{int})^2 \end{aligned} \tag{3.5}$$

We used here $\int_{\mathcal{D}[\theta]} |\theta_\omega|^2 \mathrm{e}^{-S_0} / \int_{\mathcal{D}[\theta]} \mathrm{e}^{-S_0} = G_\omega$. The second term in the numerator is the disconnected term. It is convenient to derive this expression by derivation with G_ω

$$\Sigma^{(1)}_\omega = -2\frac{\partial}{\partial G_\omega} \langle S_{int} \rangle_{S_0} = -2\frac{\partial}{\partial G_\omega} \frac{\int_{\mathcal{D}[\theta]} \mathrm{e}^{-S_0} S_{int}}{\int_{\mathcal{D}[\theta]} \mathrm{e}^{-S_0}} =$$
$$-G_\omega^{-2} \frac{\int_{\mathcal{D}[\theta]} \mathrm{e}^{-S_0} |\theta_\omega|^2 S_{int}}{\int_{\mathcal{D}[\theta]} \mathrm{e}^{-S_0}} + G_\omega^{-2} \frac{\int_{\mathcal{D}[\theta]} \mathrm{e}^{-S_0} G_\omega S_{int}}{\int_{\mathcal{D}[\theta]} \mathrm{e}^{-S_0}} \tag{3.6}$$

Where the first order Green's function is

$$G^{(1)}_\omega = G_\omega + G_\omega^2 \Sigma^{(1)}_\omega \tag{3.7}$$

In the following subsections we calculate the Green's function perturbation for the first sector $m = 0$ with $\phi_x = 0$ and for $T = 0$ for large α. We use here the convention for the Matsubara integral as $\int_\omega = T \sum_{\omega_n}$ and $\int_\tau = \int_0^\beta d\tau$, for $T = 0$ the integrals are $\int_\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega$ and $\int_\tau = \int_{-\infty}^{\infty} d\tau$.

3.2.1 Large α perturbation

To have perturbation in large α we add and subtract a dissipative term to S_0 and S_{int} respectively. The dissipation term is chosen such that in the limit of large α (which is equivalent to the large ω) the S_{int} term vanished. Here we only consider the zero winding section $m = 0$ and assume the result is flux independent, in section 3.5 this is justified numerically. The action is

$$S_0 = \frac{1}{2} \int_\omega \left[M\omega^2 + \pi\alpha |\omega| \right] |\theta_\omega|^2$$
$$S_{int} = -2 \int_{\tau,\tau'} g_{\tau,\tau'} \left(\sin^2 \left(\frac{\theta_\tau - \theta_{\tau'}}{2} \right) - \left(\frac{\theta_\tau - \theta_{\tau'}}{2} \right)^2 \right) \tag{3.8}$$

The bare Green's function is $G_\omega = M\omega^2 + \pi\alpha |\omega|$. The first term in S_{int} is

$$\langle S_{int} \rangle_{S_0} = -2 \int_{\tau,\tau'} g_{\tau,\tau'} \left\langle \sin^2 \left(\frac{\theta_\tau - \theta_{\tau'}}{2} \right) \right\rangle_{S_0} = -\int_{\tau,\tau'} g_{\tau,\tau'} (1 - \mathrm{e}^{G_{\tau,\tau'} - G_{\tau=0}}) =$$
$$\int_\tau g_\tau \mathrm{e}^{-\int_\omega (1-\cos\omega\tau) G_\omega} \tag{3.9}$$

The second term in the perturbation is a Gaussian term with $\langle S_{int}\rangle = -\alpha\frac{\pi}{2}\int_\omega |\omega| \left\langle |\theta_\omega|^2\right\rangle_0 = -\alpha\frac{\pi}{2}\int_\omega \eta\,|\omega|\,G_\omega$. Hence to 1st order

$$\begin{aligned}\Sigma^{(1)}_\omega = -2\frac{\partial}{\partial G_\omega}\langle S_{int}\rangle_{S_0} &= 2\int_\tau (1-\cos\omega\tau) g_\tau \mathrm{e}^{-\int_\omega (1-\cos\omega\tau)G_\omega} + \pi\alpha\,|\omega| \\ &= -\alpha\int_\tau \frac{1-\cos\omega\tau}{\tau^2}\mathrm{e}^{-\int_{\omega'}\frac{1-\cos\omega'\tau}{M\omega'^2+\pi\alpha|\omega'|}} + \pi\alpha\,|\omega| \end{aligned} \tag{3.10}$$

The integral in the exponent with $\omega_c = \eta/M$

$$\int_\omega \frac{1-\cos\omega\tau}{M\omega^2+\pi\alpha\,|\omega|} = \frac{1}{\pi^2\alpha}\left[\gamma + \log(\tau\omega_c) - \sin(\tau\omega_c)\mathrm{Si}(\tau\omega_c) - \cos(\tau\omega_c)\mathrm{Ci}(\tau\omega_c) + \frac{\pi}{2}\sin(\tau\omega_c)\right] =$$
$$\frac{1}{\pi^2\alpha}\left[\gamma + \log\tau\omega_c + \frac{1}{(\tau\omega_c)^2} + \mathcal{O}\big(\tau^{-4}\big)\right] \qquad \tau\omega_c \gg 1 \tag{3.11}$$

For small frequencies we can take large time terms in the exponent

$$\begin{aligned}\Sigma_1(\omega) &= -\alpha\int_\tau \frac{1-\cos\omega\tau}{\tau^2}(\tau\omega_c)^{-1/\pi\eta}\,\mathrm{e}^{-\frac{\gamma}{\pi^2\alpha}} = -\alpha \mathrm{g}_\alpha\,|\omega|^{1+1/\pi^2\alpha}\,\omega_c^{-1/\pi^2\alpha} \\ \mathrm{g}_\alpha &= 2\sin\frac{1}{2\pi\alpha}\mathrm{e}^{-\frac{\gamma}{\pi^2\alpha}}\Gamma(-1-\frac{1}{\pi^2\alpha}) > 0 \quad \text{for} \quad \alpha > 1/\pi^2\end{aligned} \tag{3.12}$$

for $\alpha < 1/\pi^2$ this integral diverges for short times (this divergent is not real, for short time the exponent is $\mathrm{e}^{-|\tau|/2M}$, meaning there is no short times divergence). The first order Green's function for $\alpha > 1/\pi^2$

$$G^{(1)}_\omega = \frac{1}{M\omega^2+\pi\alpha\,|\omega|} - \alpha\frac{\mathrm{g}_\alpha\omega^{1+1/\pi^2\alpha}\omega_c^{-1/\pi^2\alpha}}{[M\omega^2+\pi\alpha\,|\omega|]^2} + \pi\alpha\frac{|\omega|}{[M\omega^2+\pi\alpha\,|\omega|]^2} \tag{3.13}$$

expansion in large α $g_\alpha\,|\omega/\omega_c|^{1/\pi^2\alpha} = \pi - (1-\log\frac{\omega}{\omega_c})/(\pi\alpha) + \mathcal{O}(\alpha^{-2})$

$$G^{(1)}_\omega = \frac{1}{M\omega^2+\pi\alpha\,|\omega|} + \alpha\frac{|\omega|}{\pi}\frac{1-\log\frac{\omega}{\omega_c}}{[M\omega^2+\pi\alpha\,|\omega|]^2} \tag{3.14}$$

identify α^R by $\lim_{\omega\to0}\omega G_1(\omega)$

$$\frac{1}{\alpha^R} = \frac{1}{\alpha} + \frac{1}{\pi^2\alpha^2}(1-\log\frac{\omega}{\omega_c}) \tag{3.15}$$

The RG equation which was derived at [42]

$$\frac{d(1/\tilde\alpha)}{d\log\omega} = -\frac{1}{\tilde\alpha^2} \qquad \tilde\alpha = \pi^2\alpha \tag{3.16}$$

3.3 Small α perturbation

In the previous section we considered only the zero sector $m = 0$, and $\phi_x = 0$, for large α. The small α perturbation can be solved for all winding and in finite ϕ_x. The action (3.4) can be written as

$$S^{(m)} = \frac{1}{2}M\int_0^{1/T} \dot{\theta}_\tau^2 d\tau + 2\pi^2 TMm^2 + \iint_0^{1/T} g_{\tau,\tau'}^{(m)} \cos\left(\theta_\tau - \theta_{\tau'}\right)\ d\tau d\tau' \tag{3.17}$$

with $g_\tau^{(m)} = g_\tau \cos(2\pi m\tau T)$. In this expression we used the symmetry between θ_τ and $\theta_{\tau'}$ in the $\int \mathcal{D}[\theta]$ integral. The action

$$Z = \int \mathcal{D}[\theta]\mathrm{e}^{-S_0} \sum_m \mathrm{e}^{-2\pi^2 TMm^2 + 2\pi i\phi_x m}\mathrm{e}^{-\iint_0^{1/T} g_{\tau-\tau'}^{(m)} \cos(\theta_\tau - \theta_{\tau'})d\tau d\tau'} \tag{3.18}$$

$$S_0 = \frac{1}{2}M\int_0^{1/T} \dot{\theta}_\tau^2\ d\tau$$

First order perturbation in α where the bare Green's function is $G_\omega = 1/M\omega^2$ and in time domain $G_\tau = |\tau|/2M$.

$$Z = \int_{\mathcal{D}[\theta]} \mathrm{e}^{-S_0} \sum_m \mathrm{e}^{-2\pi^2 TMm^2 + 2i\pi m\phi_x} \left[1 - \iint_0^{1/T} g_{\tau-\tau'}^{(m)} \cos\left(\theta_\tau - \theta_{\tau'}\right) d\tau d\tau'\right] \tag{3.19}$$

The first order perturbation of the Green's function

$$G_\omega^{(1)} = G_\omega + \frac{\sum_m \mathrm{e}^{-2\pi^2 TMm^2 + 2i\pi m\phi_x} \int_{\mathcal{D}[\theta]} \mathrm{e}^{-S_0} \int_{\tau,\tau'} g_{\tau-\tau'}^{(m)} \cos\left(\theta_\tau - \theta_{\tau'}\right) \left[G_\omega - |\theta_\omega|^2\right]}{\sum_m \mathrm{e}^{-2\pi^2 TMm^2 + 2i\pi m\phi_x} \int_{\mathcal{D}[\theta]} \mathrm{e}^{-S_0[\theta]}} =$$

$$G_\omega + G_\omega^2 \frac{\sum_m \mathrm{e}^{-2\pi^2 TMm^2 + 2i\pi m\phi_x} \Sigma_\omega^{(m)}}{\sum_m \mathrm{e}^{-2\pi^2 TMm^2 + 2i\pi m\phi_x}} \tag{3.20}$$

where $\Sigma_\omega^{(m)}$ is similar to the previous section where the bare Green's function is

$$\Sigma_\omega^{(m)} = 2\int_{-1/2T}^{1/2T} (1 - \cos\omega\tau) \cos(2\pi m\tau T) g_\tau \mathrm{e}^{-\int_\omega (1-\cos\omega\tau) G_\omega}\ d\tau \tag{3.21}$$

We solve this integral considering the finite temperature. In Matsubara formalism the integral for finite T turn to the summation $\int_\omega f_\omega = T\sum_{k=-\infty}^{\infty} f_{\omega_k}$, with $\omega_k = 2\pi kT$. The sum in the exponent

$$T\sum_{\omega_k} G_{\omega_k}(1-\cos\omega_k\tau) = \frac{T}{M}\sum_{k\neq 0}\frac{1-\cos 2\pi T\tau k}{4\pi^2T^2k^2} = \frac{T}{M}\sum_{k}\frac{1-\cos 2\pi T\tau k}{4\pi^2T^2k^2} - \frac{T}{2M}\tau^2 =$$
$$\frac{T}{M}\iint_{\tau} 2\pi\delta(2\pi T\tau) - \frac{T}{2M}\tau^2 = |\tau|/2M - T\tau^2/2M \tag{3.22}$$

The expression for the first order perturbation

$$\Sigma^{(m)}_{\omega_k} = -2\alpha\int_0^{1/2T} d\tau\frac{\pi^2T^2}{\sin^2\pi\tau T}\cos 2\pi m\tau T(1-\cos\omega_k\tau)\mathrm{e}^{-\tau(1-T\tau)/(2M)}$$
$$G^{(1)}_{\omega_k} = G_{\omega_k} + G^2_{\omega_k}\frac{\sum_m \mathrm{e}^{-2\pi^2TMm^2+2i\pi m\phi_x}\Sigma^{(m)}_{\omega_k}}{\sum_m \mathrm{e}^{-2\pi^2TMm^2+2i\pi m\phi_x}} = G_{\omega_k} + G^2_{\omega_k}\Sigma^{(1)}_{\omega_k} \tag{3.23}$$

The Poisson summation formula $\sum_{m=-\infty}^{\infty} g(m) = \int_{-\infty}^{\infty}\sum_{K=-\infty}^{\infty}\mathrm{e}^{2\pi i\phi K}d\phi$

$$\frac{\sum_m \mathrm{e}^{-2\pi^2TMm^2+2i\pi m\phi_x}\cos(2\pi m\tau T)}{\sum_m \mathrm{e}^{-2\pi^2TMm^2+2i\pi\phi_x m}} = \frac{1}{2}\sum_{\sigma=\pm}\frac{\sum_K\int_\phi \mathrm{e}^{2\pi i\phi K}\mathrm{e}^{-2\pi^2TM\phi^2+2i\pi\phi\phi_x}\cos(2\pi\phi\tau T)}{\sum_K\int_\phi \mathrm{e}^{2\pi i\phi K}\mathrm{e}^{-2\pi^2TM\phi^2+2i\pi\phi\phi_x}} =$$
$$\frac{1}{2}\sum_{\sigma}\frac{\sum_K \mathrm{e}^{-\frac{(K+\phi_x+\sigma\tau T)^2}{2TM}}}{\sum_K \mathrm{e}^{-\frac{(K+\phi_x)^2}{2TM}}} = \mathrm{e}^{-\frac{\tau^2T}{2M}}\frac{\sum_K \mathrm{e}^{-\frac{(K+\phi_x)^2}{2TM}}\cosh(\tau(K+\phi_x)/M)}{\sum_K \mathrm{e}^{-\frac{(K+\phi_x)^2}{2TM}}} \tag{3.24}$$

Notice that $\mathrm{e}^{-\tau^2T/(2M)}$ cancel out.

$$\Sigma^{(1)}_{\omega_k} = -2\alpha\int_0^{1/2T}\frac{\pi^2T^2}{\sin^2\pi\tau T}(1-\cos\omega_k\tau)\frac{\sum_{K,\sigma=\pm}\mathrm{e}^{-\frac{(K+\phi_x)^2}{2TM}}\mathrm{e}^{\frac{-\tau}{M}(\frac{1}{2}+K+\sigma\phi_x)}}{2\sum_K \mathrm{e}^{-\frac{(K+\phi_x)^2}{2TM}}} \tag{3.25}$$

3.3.1 $0 \le \phi_x < 1/2$

For the limit $T \to 0$ the dominant term for this flux is $K = 0$

$$\Sigma^{(1)}_{\omega_k} = -2\alpha\int_0^{\frac{1}{2T}}\frac{(T\pi)^2}{\sin^2\pi T\tau}(1-\cos 2\pi kT\tau)\mathrm{e}^{\frac{-\tau}{2M}}\cosh\frac{\tau\phi_x}{2M} = -2\alpha\frac{\omega_k^2}{1-4\phi_x^2} + \mathcal{O}\big(\omega_k^4\big) \tag{3.26}$$

This means that the perturbative correction does not create a dissipative term, only a correction to the mass. For $\phi_x = 0$ the result is as that found for $m = 0$ in the previous subsection. We can also see that for large ω_k $\Sigma(\omega_k) = \pi\alpha|\omega_k|$ as expected.

3.3.2 $\phi_x = 1/2$

For $\phi_x = 1/2$ the contribution from $K = 0, -1$ terms are the dominant ones. Their contribution is the same

$$\frac{\sum_K \mathrm{e}^{-\frac{(K+\phi_x)^2}{2TM}} \left(\mathrm{e}^{\frac{-\tau}{M}(\frac{1}{2}+K+\phi_x)} + \mathrm{e}^{\frac{-\tau}{M}(\frac{1}{2}-K-\phi_x)}\right)}{2\sum_K \mathrm{e}^{-\frac{(K+\phi_x)^2}{2M}}} = \frac{1}{2} + \mathcal{O}\big(\mathrm{e}^{-\tau/M}\big) \tag{3.27}$$

With that the self energy expression is

$$\begin{aligned} \Sigma^{(1)}_{\omega_k} &= -\frac{\alpha}{2}\int_{-1/2T}^{1/2T} \frac{(\pi T)^2}{\sin^2 \pi\tau T}(1-\cos\omega_k\tau) = \int_{-1/2T}^{1/2T} g_\tau(1-\cos\omega_k\tau) = \\ g_{\omega_k=0} - g_{\omega_k} &= -g_{\omega_k} = -\alpha\frac{\pi}{2}\left|\omega_k\right| \end{aligned} \tag{3.28}$$

The perturbative Green's function is

$$G^{(1)}_{\omega_k} = G_{\omega_k} + G^2_{\omega_k}\Sigma^{(1)}_{\omega_k} = \frac{1}{M\omega_k^2} + \frac{\Sigma_1(\omega_k)}{M^2\omega_k^4} \tag{3.29}$$

$$\Sigma^{(1)}_{\omega_k} = -\alpha \begin{cases} 2\omega_k^2 & \phi_x = 0 \\ \frac{2}{1-4\phi_x^2}\omega_k^2 & 0 < \phi_x < 1/2 \\ \frac{\pi}{2}\left|\omega_k\right| & \phi_x = 1/2 \end{cases} \quad + \text{higher order of } \omega_k$$

This perturbation does not show the expected result, by which we expected G_ω to have a dissipative term after averaging over the flux $1/\eta^R_{\phi_x} = \int_0^1 K(\phi_x)d\phi_x$. We do find a dissipative term at flux half as expected but this term has zero weight upon averaging over flux.

We expect $K_\omega = \omega_k^2 G_{\omega_k} = K_0(\phi_x) + \left|\omega_k\right| K_1(\phi_x)$. Hence the dissipative form is obtained only at $\phi_x = \frac{1}{2}$ with $K_1 = \frac{\pi}{2\omega_k^2}$. The singularity implies $K_1 \sim \delta(\phi_x - \frac{1}{2})$ in fact by resummation of the small α perturbation [27] found that $K_1 \sim T\delta(\phi_x - \frac{1}{2})$ at $T \to 0$, i.e. a renormalized $\eta^R \sim T \to 0$

3.4 Monte Carlo (MC) simulation

In this subsection we describe the numerical method used to solve the action . This model was studied intensively using MC algorithm focusing on the effective mass [9, 19, 20], but no work has been done to calculate the effective α. It should be noted that for the related problem of particle in periodic potential an effective dissipation was calculated using MC simulations [43, 44], where it is well known that at $\alpha_c = 1/(2\pi^2)$ there is a phase transition between a dissipative regime ($\alpha^R \to \alpha$) for $\alpha < \alpha_c$ and a localized regime ($\alpha^R \to \infty$) for $\alpha > \alpha_c$ [16, 45].

3.4.1 Model

We solve the model where the Matsubara partition function for particle on a ring in CL environment is

$$
\begin{aligned}
Z &= \sum_{m=-\infty}^{\infty} \mathrm{e}^{-2\pi^2 m^2 MT + 2\pi i m\phi_x} \int_{\mathcal{D}[\theta]} \mathrm{e}^{-S^{(m)}} \\
S^{(m)} &= \frac{1}{2} MR^2 \int_0^{1/T} \dot{\theta}_\tau^2 d\tau + \iint_0^{1/T} g^{(m)}_{\tau-\tau'} \cos\left(\theta_\tau - \theta_{\tau'}\right) \\
g^{(m)}_{\omega_k} &= \frac{\alpha\pi}{4}\left(|\omega_{k+m}| + |\omega_{k-m}|\right) \\
g^{(m)}_\tau &= T \sum_{k=-\infty}^{\infty} g^{(m)}_{\omega_k} \mathrm{e}^{i\tau\omega_k} = -\frac{\alpha}{2}\frac{(\pi T)^2}{\sin^2 \pi T\tau} \cos(2\pi m T\tau) \qquad \tau \neq 0
\end{aligned}
\tag{3.30}
$$

m is the winding number. The $g_{\tau=0}$ term has a positive divergence so that $g_{\omega_{k=0}} = \int_0^{1/T} g_\tau = 0$. We wish to calculate numerically the Green's function

$$
\tilde{K}_\tau = \left\langle \dot{\theta}^{(m)}_t \dot{\theta}^{(m)}_{\tau+t} \right\rangle = \left\langle \dot{\theta}_t \dot{\theta}_{\tau+t} \right\rangle + 4\pi^2 T^2 (\left\langle m^2 \right\rangle - \left\langle m \right\rangle^2) \tag{3.31}
$$

with $\theta^{(m)}_\tau = \theta_\tau + 2\pi m T\tau$, or alternatively in frequency space

$$
\tilde{K}_{\omega_k} = T\omega_k^2 \left\langle |\theta_{\omega_k}|^2 \right\rangle + 4\pi^2 T^2 (\left\langle m^2 \right\rangle - \left\langle m \right\rangle^2)\delta(\omega_k) \equiv C_{\omega_k} + \frac{1}{M^*}\delta_{k,0} \tag{3.32}
$$

With $T\delta(\omega_k) = \delta_{k,0}$. The first term $K(\omega_k)$ is the correlation for $k \neq 0$. The second term is the curvature.

$$
\frac{1}{M^*} = \frac{\partial^2 \mathcal{F}}{\partial \phi_x^2} = -T\frac{\partial^2 \log Z}{\partial \phi_x^2} = 4\pi^2 T (\left\langle m^2 \right\rangle - \left\langle m \right\rangle^2) \tag{3.33}
$$

3.4.2 Time discretization of the action

The numerical realization of the action requires discretization of the action by dividing the time integral to N trotter segments with $\Delta\tau = 1/(TN)$. Then $\tau = \Delta\tau j,\ j = 0, 1, ...N-1$. the variable θ_j is the time discretization of θ_τ. Equivalently in frequency domain the number of modes ω_k is N where $\omega_k = 2\pi kT$ and k an integer $k = 0, 1, ...N-1$.

For numerical time considerations it is more suitable to work with an action written in Fourier space. Realization of the action in the time domain requires a computation time of $\mathcal{O}(N^2)$, where realization in frequency space requires numerical time of $\mathcal{O}(N \log N)$.

The Conventions of Fast Fourier algorithm are that the terms $k > N/2$ are considered instead of negative frequencies. θ_j is real means $\theta_k = \theta^*_{N-k}$. The frequencies are $\omega_k = 2\pi Tk$ for $k < N/2$ and for $k \geq N/2$ $\omega_k = \omega_{N-k}$, so the action is symmetric. Note that $\theta_{k=0}$ does not contribute to the action.

$$S^{(m)} = S_0 + S^{(m)}_\alpha = T\Delta\tau \sum_{k=0}^{N-1} \left[\frac{M\Delta\tau}{2} \omega_k^2 \left|\theta_k\right|^2 + g_k^{(m)} \left|\psi_k\right|^2 \right] \tag{3.34}$$

Where $g_k^{(m)}$ the discretize version of g_{ω_k}

$$g_k^{(m)} = \frac{\pi}{2} \alpha \Delta\tau \times \begin{cases} |\omega_k| = 2\pi T\,|k| & |\omega_k| > |\omega_m| \\ |\omega_m| = 2\pi T\,|m| & |\omega_k| < |\omega_m| \end{cases} \tag{3.35}$$

The partition function is

$$Z = \mathrm{e}^{-\beta\mathcal{F}} = \mathrm{e}^{-\frac{MT}{2}\sum_k \omega_k^2 |\theta_k|^2 \Delta\tau^2} \sum_{m=-\infty}^{\infty} \mathrm{e}^{-2\pi^2 m^2 MT} \mathrm{e}^{-2\pi i m \phi_x} \mathrm{e}^{-T\Delta\tau \sum_k g_k^{(m)} |\psi_k|^2} \tag{3.36}$$

We use here the Fast Fourier conventions for discrete transformation

$$O_k = \sum_{j=0}^{N-1} \mathrm{e}^{2\pi i \frac{jk}{N}} O_j \quad ; \quad O_j = \frac{1}{N} \sum_{k=0}^{N-1} \mathrm{e}^{-2\pi i \frac{jk}{N}} O_k \tag{3.37}$$

Setting $\Delta\tau/M = 1/(TNM)$ as the dimensionless time unit. For convenience we choose $M = 1$ in all simulations, which left us with three free parameters in the model, α, N and T.

3.4.3 Green's function

We are interested in the velocity velocity Green's function of Eq. (3.31)

$$\tilde{K}_\tau = \left\langle \dot{\theta}_t^{(m)} \dot{\theta}_{\tau+t}^{(m)} \right\rangle = \left\langle \dot{\theta}_t \dot{\theta}_{\tau+t} \right\rangle + 4\pi^2 T^2 (\langle m^2 \rangle - \langle m \rangle^2) \tag{3.38}$$

Where $\langle \cdot \rangle$ is understood as an average over both the action and over the time domain t. The Green's function for the periodic part in is $C_\tau = \left\langle \dot{\theta}_t \dot{\theta}_{\tau+t} \right\rangle$. This function in frequency space is found by the Wiener-Khinchin theorem, where $\dot{\theta}_\omega$ is the Fourier Transform of $\dot{\theta}_\tau$

$$\begin{aligned} C_\omega = \int_\tau C_\tau \mathrm{e}^{i\omega\tau} = \int_\tau \mathrm{e}^{i\omega\tau} \int_{\omega_1,\omega_2} \left\langle \dot{\theta}_{\omega_1} \dot{\theta}_{\omega_2} \right\rangle \mathrm{e}^{-i\omega_1 t - i\omega_2(\tau - t)} = \\ \frac{\Delta\omega}{2\pi} \int_t \int_\tau \mathrm{e}^{i\omega\tau} \int_{\omega_1,\omega_2} \left\langle \dot{\theta}_{\omega 1} \dot{\theta}_{\omega 2} \right\rangle \mathrm{e}^{-i\omega_1 t - i\omega_2(\tau - t)} = \frac{\Delta\omega}{2\pi} \left\langle |\dot{\theta}_\omega| \right\rangle \end{aligned} \tag{3.39}$$

with the time span $1/T$, $\Delta\omega = 2\pi T$ and we get $C_\omega = T \left\langle \left| \dot{\theta}_\omega \right|^2 \right\rangle$. In discrete form the time correlation function is $C_{j_1,j_2} = \left\langle \dot{\theta}_{j1} \dot{\theta}_{j2} \right\rangle$, C_k is the Green's function in discrete frequency form, with the Wiener-Khinchin theorem it is

$$\begin{aligned} C_k = \sum_{j=0}^{N-1} C_j \mathrm{e}^{2\pi i \frac{jk}{N}} = \frac{1}{N^2} \sum_j \mathrm{e}^{2\pi i \frac{jk}{N}} \sum_{k_1,k_2} \left\langle \dot{\theta}_{k_1} \dot{\theta}_{k_2} \right\rangle \mathrm{e}^{-2\pi i \frac{j_1 k_1}{N} - 2\pi i \frac{(j_1 - j) k_2}{N}} = \\ \frac{1}{N^3} \sum_{j_1} \sum_j \mathrm{e}^{2\pi i \frac{jk}{N}} \sum_{k_1,k_2} \left\langle \dot{\theta}_{k_1} \dot{\theta}_{k_2} \right\rangle \mathrm{e}^{-2\pi i \frac{jk_1}{N} - 2\pi i \frac{(j_1 - j) k_2}{N}} = \frac{1}{N} \left\langle \left| \dot{\theta}_k \right|^2 \right\rangle \end{aligned} \tag{3.40}$$

where $\dot{\theta}_k$ is the discrete Fourier Transform of $\dot{\theta}_j = (\theta_{j+1} - \theta_j)/\Delta\tau$. The continuum $\dot{\theta}_\omega$ and the discrete $\dot{\theta}_k$ are related by

$$\dot{\theta}_\omega = \int_\tau \dot{\theta}_\tau \mathrm{e}^{i\omega\tau} = \Delta\tau \sum_{j=0}^{N-1} \dot{\theta}_j \mathrm{e}^{2\pi i \frac{jk}{N}} = \Delta\tau \dot{\theta}_k \tag{3.41}$$

and the Green's functions are related by

$$C_\omega = T \left\langle \left| \dot{\theta}_\omega \right|^2 \right\rangle = T\Delta\tau^2 \left\langle \left| \dot{\theta}_k \right|^2 \right\rangle = \frac{\Delta\tau}{N} \left\langle \left| \dot{\theta}_k \right|^2 \right\rangle = \Delta\tau C_k \tag{3.42}$$

The full correlation functions, including also the winding part of Eq. (3.31)

$$\tilde{K}_\omega = T\left\langle \left|\dot{\theta}_\omega\right|^2 \right\rangle + 4\pi^2 T(\langle m^2\rangle - \langle m\rangle^2)\delta_{k,0} = T\left\langle \left|\dot{\theta}_\omega\right|^2 \right\rangle + 4\pi T^2(\langle m^2\rangle - \langle m\rangle^2)\delta(\omega)$$

$$\tilde{K}_k^{(m)} = \frac{1}{N}\left\langle \left|\dot{\theta}_k\right|^2 \right\rangle + 4\pi^2\frac{T}{\Delta\tau}(\langle m^2\rangle - \langle m\rangle^2)\delta_{k,0} \tag{3.43}$$

with $\dot{\theta}_{k=0} = 0$ due to the periodicity, and $\delta_{k,0} = T\delta(\omega)$.

3.4.4 Monte Carlo (MC) Algorithm

In this section we outline the numerical algorithm used to calculate an expectation value $Q = \langle Q\rangle_S$ where the average is done with respect to an action S, the partition function is $Z = \int \mathcal{D}[\mathbf{x}]\mathrm{e}^{-S[\mathbf{x}]}$ and $\mathbf{x}$ are action variables.

The MC algorithm goes as follows [46]: one starts with an arbitrary values for the variables $\mathbf{x}$, and runs a long series of steps of a Markov chain. In this chain of steps one moves between different states of the variables $\mathbf{x}$ so as to approach equilibrium. In any state state μ of the system the expectation value Q_μ is calculated, the final result for the expectation value is $Q = \frac{1}{M}\sum_{i=1}^{M} Q_{\mu_i}$, where M is the number of states average over.

In order for the system to approach equilibrium and remain there two conditions are imposed on the transition probability between states. The first is ergodicity, the requirement that the Markov process can reach any state from any other state. The second is detailed balance that requires the ratio between $P_{\mu\to\nu}$, the probability for a transition between state μ and state ν, and $P_{\nu\to\mu}$ the probability for a transition between state ν and state μ, to equal the ratio between the probability to be in state ν (or the weight of state ν) which is $\mathrm{e}^{-S[\mathbf{x}_\nu]}$ to the probability to be in state μ, explicitly

$$\frac{P_{\mu\to\nu}}{P_{\nu\to\mu}} = \mathrm{e}^{-(S[\mathbf{x}_\nu]-S[\mathbf{x}_\mu])} \tag{3.44}$$

The probability $P_{\mu\to\nu}$ for transition between states is composed of two terms, a selection probability that $g_{\mu\to\nu}$ is the probability the $\mu \to \nu$ move is suggested, and an acceptance probability $A_{\mu\to\nu}$. The transition probability is then $P_{\mu\to\nu} = g_{\mu\to\nu}A_{\mu\to\nu}$.

3.4.5 Metropolis algorithm

There are several possible choices for $P_{\mu\to\nu}$ that obey the ratio of Eq. (3.44). The Metropolis algorithm is a commonly used choice, in a standard Metropolis algorithm the MC step changes the state $\mu \to \nu$ by taking in each step a random index $j \in [1, N]$ and a suggests update $\theta_j \to \theta_j \pm \delta\theta_j$ with probability $g_{\mu\to\nu} = 1$. Then the update is accepted with the transition probability

$$A_{\mu\to\nu} = P_{\mu\to\nu} = \begin{cases} \mathrm{e}^{-(S[\mathbf{x}_\nu]-S[\mathbf{x}_\mu])} & S[\mathbf{x}_\nu] - S[\mathbf{x}_\mu] > 0 \\ 1 & \text{otherwise} \end{cases}. \tag{3.45}$$

and the suggested update amplitude $\delta\theta_j$ is determined dynamically in the simulation to produce an acceptance ratio of 1/2. For the partition function in our model we define an effective action that contains all windings

$$Z = \sum_{m=-\infty}^{\infty} \mathrm{e}^{-2\pi^2 m^2 MT + 2\pi i m \phi_x} \int \mathcal{D}[\theta] \mathrm{e}^{-S^{(m)}} \equiv \int \mathcal{D}[\theta] \mathrm{e}^{-S_{eff}[\theta]}. \tag{3.46}$$

Following an idea from [43] our algorithm is the following. In each numerical step we create a set of θ_k using the selection probability

$$g_{\mu\to\nu} = \mathrm{e}^{-S_0[\theta_\mu]} = \mathrm{e}^{-\frac{1}{2}T\Delta\tau^2 \sum_{k=1}^{N-1} M\omega_k^2 |\theta_k|^2} \tag{3.47}$$

Meaning we choose random set of θ_k $k = 1, 2..N/2$ taken from an exponential distribution with mean $2/(T\Delta\tau^2 M\omega_k^2)$ and a uniform random phase and $\theta_{N-k} = \theta_k^*$. The acceptance probability is taken as in the Metropolis algorithm excluding the Gaussian kinetic mass term,

$$A_{\mu\to\nu} = \min\left[1, \frac{\sum_m \mathrm{e}^{-2\pi^2 m^2 MT + 2\pi i m \phi_x} \mathrm{e}^{-S_\alpha^{(m)}[\theta_\mu]}}{\sum_m \mathrm{e}^{-2\pi^2 m^2 MT + 2\pi i m \phi_x} \mathrm{e}^{-S_\alpha^{(m)}[\theta_\nu]}}\right] \tag{3.48}$$

so that Eq. (3.44) is satisfied.

3.4.6 The sign problem

A problem arises here since the Metropolice algorithm requires the weights $e^{-S_{eff}[\theta]}$ to be non-negative. In our model the winding summation $\sum_m e^{-2\pi^2 m^2 MT + 2\pi i m\phi_x} e^{-S_\alpha^{(m)}[\theta_\mu]}$ can be negative for $\phi_x \neq 0$. In particular near $\phi_x = 1/2$ the alternating sign means that the number of states with negative weight approaches the number of states with positive weight as the temperature decreases. This is due to the factor $e^{-2\pi^2 m^2 MT}$, where for large T the contribution to the summation is restricted to the $m = 0$ sector but as the temperature decreases more winding sectors contribute to the value of Z.

A standard method [47] to calculate an expectation value Q when the weight P can be negative is to replace the calculation of the values Q_μ with respect to weights P_μ by the values of $Q_\mu S_\mu |P_\mu|^{1-\kappa}$ with respect to the weights $|P_\mu|^\kappa$ where S_μ is the sign of P_μ. Then the expectation value of $\langle Q \rangle_P$ is for any real κ

$$\langle Q \rangle_P = \frac{\sum_\mu P_\mu Q_\mu}{\sum_\mu P_\mu} = \frac{\sum_\mu S_\mu |P_\mu|^{1-\kappa} |P_\mu|^\kappa Q_\mu}{\sum_\mu S_\mu |P_\mu|^{1-\kappa} |P_\mu|^\kappa} = \frac{\frac{\sum_\mu S_\mu |P_\mu|^{1-\kappa} |P_\mu|^\kappa Q_\mu}{\sum_\mu |P_\mu|^\kappa}}{\frac{\sum_\mu S_\mu |P_\mu|^{1-\kappa} |P_\mu|^\kappa}{\sum_\mu |P_\mu|^\kappa}} = \frac{\langle S \cdot Q \rangle_{|P|^\kappa}}{\langle S \rangle_{|P|^\kappa}} \quad (3.49)$$

This method solves the issue of negative weights but a different problem, the infamous 'sign problem', arises. In our simulation we use the conventional scheme with $\kappa = 1$ where the average sign $\langle S \rangle_{|P|}$ decreases exponentially with the numerical size N or equivalently with $1/T$ [47] which means that the standard deviation of the expression increases exponentially as N increases, making the numerical results unreliable for large N values.

3.4.7 Effective Mass

The zero frequency term $\tilde{K}_{k=0} = \frac{1}{M^*} = 4\pi^2 \frac{T}{\Delta\tau}(\langle m^2 \rangle - \langle m \rangle^2)$ defines the effective mass of the particle and is found by calculating the winding distribution Z_m during the simulation.

$$\frac{1}{M^*} = 4\pi^2 T(\langle m^2 \rangle - \langle m \rangle^2) = 4\pi^2 T \left(\frac{\sum_m m^2 Z_m}{\sum_m Z_m} - \left(\frac{\sum_m m Z_m}{\sum_m Z_m} \right)^2 \right)$$

$$Z_m = e^{-2\pi^2 m^2 MT} e^{-2\pi i \phi_x m} e^{-\frac{T}{2}\sum_k \pi\alpha|\omega_k|^{(m)}|\psi_k|^2 \Delta\tau^2} \quad (3.50)$$

Note that second term $\langle m \rangle$ is imaginary duo to the $\mathrm{e}^{2\pi i \phi_x m}$ in Z_m, hence the $-\langle m \rangle^2$ term is positive. The first term $\langle m^2 \rangle$ can be either negative or positive, it reduces as ϕ_x approaches 1/2. Note that for some values of ϕ_x the term $1/M^*(\phi_x)$ must be negative as the integral over the flux is $\int_0^1 \frac{1}{M^*(\phi_x)} d\phi_x = 0$.

The effective mass was calculated in numerous works in the past for this model and for other related models [9, 19, 20]. An exponential decay of $1/M^*(\phi_x = 0)$ as α increases is well established.

3.5 Numerical results and discussion

In Fig.6 the MC data for $\tilde{K}_{\omega_k}$ for the three values $N = 100, 200, 400$ and $\alpha = 1/(2\pi^2)$ are presented as well as for $N = 100$ and $\alpha = 5$. Dots are the data for fluxes $\phi_x = 0, 0.2, 0.4, 0.5$ (Blue, Green, Red, Cyan) green lines are the small α perturbation from Eq. (3.25). The red line is the first order large α perturbation of Eq. (3.13). The zero frequency ($k = 0$) data points for $1/M^*(\phi_x)$ are seen only for the small flux data, for larger flux data their value is out of the figure range. The numerical results show that for small N (high T) we get the perturbative results for both small and large values of α. However for the small α value (upper and lower left figures) as we lower the temperature (increase N), the result for large flux become unreliable and the data becomes flux independent. This result is probably caused by the numerical sign problem. We would expect that for large N values the correlation function, especially close to the degeneracy point of $\phi_x = \frac{1}{2}$, will become linear for small frequencies. That linearity, should we be able to achieve it for a low enough temperature, would define the effective dissipation value α^R, its ϕ_x integration would give η^R, the subject of our work.

The bottom right figure demonstrates that for large α values the results are flux independent and fits the perturbation of Eq. (3.13)

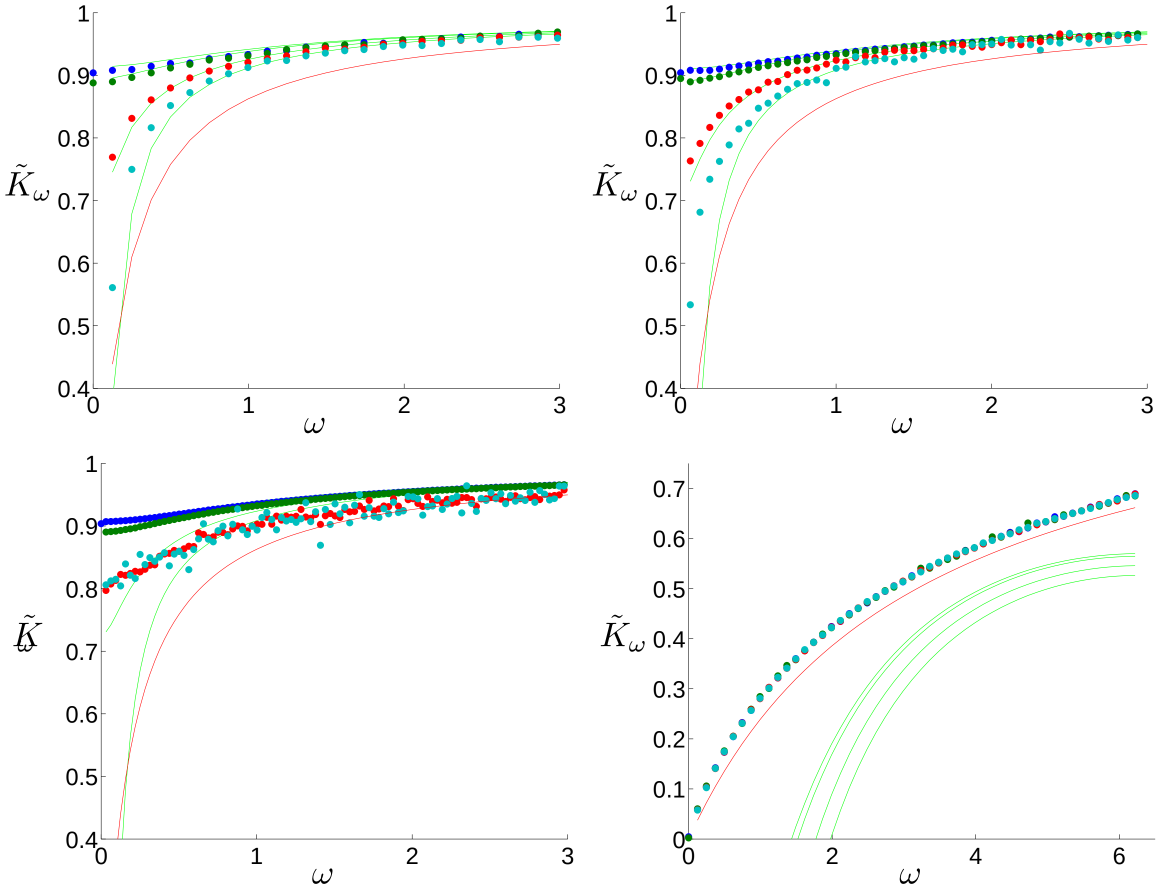

Figure 6: **Upper left panel**: MC data for $N = 100$ with $\alpha = 1/(2\pi^2)$ $\Delta\tau = 1/2$. **Upper right panel**: MC data $N = 200$ with the same parameter set. **Lower left panel**: MC data for $N = 400$ with the same parameter set. **Lower right panel**: MC data $N = 100$ with $\alpha = 5$. All data points fall on the same curve, meaning the data is flux independent.

4 Winding of planar gaussian processes

Motivated by our study of quantum noise in a ring geometry in this section we study the behavior of a general correlated noise (or process) in a two dimensional plane. The focus is the study of the angle ϕ_t of the noise $\xi_t = \xi_t^x + i\xi_t^y = |\xi_t|\, \mathrm{e}^{i\phi_t}$ and its relation to the correlation $C_{t,t'}$ of the ξ_t's. The work in this section was published in [1].

4.1 Introduction and model

The winding of planar random processes has been studied for a while. These are of interest for the physics of polymers [48–51], flux lines in superconductors [52, 53] and quantum Hall effect [54, 55]. Recently there was revived interest in winding properties of processes described by Schramm-Loewner Evolutions (SLE) [56, 57], such as the loop erased random walk [58].

The aim of this section is to study the winding of a very general continuous-time gaussian process $\xi_t = \xi_t^x + i\xi_t^y$ in the complex plane with arbitrary correlations in time. The only restriction, mainly to avoid cumbersome formula, is that the measure is rotationally invariant around the origin 0 and the winding angle ϕ_t is measured around point 0, i.e. $\xi_t = r_t e^{i\phi_t}$ where ϕ_t is a continuous real function of time and $r_t = |\xi_t|$. The process is thus centered $\langle \xi_t \rangle = 0$ and fully characterized by its two-time correlation function:

$$\langle \xi_t^i \xi_{t'}^j \rangle = \delta_{ij} C_{tt'} \tag{4.1}$$

with $i, j = x, y$, equivalently $\langle \xi_t \xi_{t'} \rangle = 0$ and $\langle \xi_t^* \xi_{t'} \rangle = 2C_{tt'}$. The most general form would be $\langle \xi_t^* \xi_{t'} \rangle = 2(C_{tt'} + iA_{tt'})$ but we also assume reflection symmetry which forbids the antisymmetric term $\epsilon_{ij} A_{tt'}$ in (4.1) with $\epsilon_{12} = -\epsilon_{21} = 1$. We use the notation $c_{tt'} = C_{tt'}/\sqrt{C_{tt} C_{t't'}}$ with $c_{tt} = 1$ and from Cauchy-Schwartz inequalities $|c_{tt'}| \leq 1$. Some particular cases are (i) stationary process $C_{tt'} = C(t - t')$ and one defines $c_{t0} = c(t) = C(t)/C(0)$ (ii) process with stationary increments $C_{tt'}^{(1,1)} := \partial_t \partial_{t'} C_{tt'} = C_2(t - t')$ (here and below we adopt the following definition for partial derivatives $\partial_t C_{tt'} = C_{tt'}^{(1,0)}$ etc..). Normalizability of the Gaussian measure requires that these functions have positive Fourier transforms $\tilde{c}(\omega) \geq 0$ and $\tilde{C}_2(\omega) \geq 0$. We restrict to a process which everywhere below we call "smooth", meaning - by definition here - ξ_t differentiable at least once, i.e. $C_{tt}^{(1,1)}$ exists (equal $-C''(0)$ for a stationary process). For such a smooth process, $\langle \dot{\xi}_t^i \xi_t^i \rangle = C_{tt}^{(1,0)} = C_{tt}^{(0,1)} = \frac{1}{2}\partial_t C_{tt}$, which vanishes if furthermore the process is stationary.

This model corresponds to our Langevin equation (2.33) with $E = 0$ where the correlation matrix is that of quantum CL noise $\tilde{C}(\omega) = \omega \coth(\hbar\omega/2T)$ (here we use the short time cutoff τ_0 in the form $\tilde{C}(\omega) = |\omega| e^{-\tau_0|\omega|}$, i.e $C(\tau) = \frac{1}{\pi}\frac{\tau_0^2 - \tau^2}{(\tau_0^2+\tau^2)^2}$) and in the limit $\eta \to 0$ (and $m \to 0$).

$$
\begin{aligned}
& m\ddot{\theta}_t + \eta\dot{\theta}_t = \xi_t^x \cos\theta_t + \xi_t^y \sin\theta_t = -|\xi_t| \sin(\theta_t - \phi_t) \\
& \sin(\theta_t - \phi_t) = 0 \quad \text{ if } \eta \to 0
\end{aligned} \tag{4.2}
$$

Then the angle θ_t is pinned to the noise phase ϕ_t, which is being studied now. We have seen that the Langevin description is valid only at large η, hence the present study does not correspond to our quantum case. Yet it has its own relevance as mentioned above.

The outline of the section is as follows. In section 4.2 we study single time quantities. The distribution of angular velocity is obtained. In section 4.3 we study the periodized winding probability distribution which is easier than the full one. The correlations of $\exp(in\phi_t)$ are obtained analytically for integer n, and studied numerically also for non-integer n. In Section 4.4 we obtain a closed formula for the variance of the winding angle as a function of the matrix $C_{tt'}$. We show that for most stationary processes the winding angle exhibits diffusion at large time and we obtain the diffusion coefficient, we also study non-stationary processes.Finally in Section 4.5 the variance of the algebraic area is obtained. Most results are tested numerically.

4.2 Single time quantities

Single time quantities are easily extracted from the Gaussian distribution $\sim d^2\xi_t e^{-|\xi_t|^2/(2C_{tt})}$ performing change of variables. Everywhere below we consider $d^2\xi = d\xi d\xi^* = d\xi^x d\xi^y$.

The modulus is distributed as $P(r_t)dr_t$ with $P(r) = \frac{r}{C_{tt}} e^{-r^2/(2C_{tt})}$, hence the probability to be within $r_t < \epsilon$ near the center vanishes as $\epsilon^2/(2C_{tt})$. To compute the distribution of the angular velocity $\dot{\phi}_t$ one uses that $X_t = (\xi_t, \dot{\xi}_t)$ is gaussian with measure $\frac{d^2\xi_t d^2\dot{\xi}_t}{(2\pi)^2} \det(M) e^{-\frac{1}{2}X^* M X}$ and correlation matrix $M^{-1} = ((C_{tt}, C_{tt}^{(1,0)}), (C_{tt}^{(0,1)}, C_{tt}^{(1,1)}))$. Let us denote $\dot{\xi}_t = \alpha_t \xi_t$ with $\alpha_t = \dot{r}_t/r_t + i\dot{\phi}_t$. Here we have requested a smooth process.The measure becomes $\frac{d^2\xi_t d^2\alpha_t}{(2\pi)^2} |\xi_t|^2 \det(M) e^{-\frac{1}{2}\beta|\xi_t|^2}$ where $\beta = (1, \alpha_t^*) M (1, \alpha_t)$. Integration over ξ_t yields the joint distribution $P(\dot{\rho}_t, \dot{\phi}_t) d\dot{\rho}_t d\dot{\phi}_t$, with $\rho_t = \ln r_t$, equal to:

$$\frac{d\dot{\rho}_t d\dot{\phi}_t}{\pi} \frac{C_{tt} C_{tt}^{(1,1)}}{(C_{tt}^{(1,1)} - 2C_{tt}^{(1,0)} \dot{\rho}_t + C_{tt}(\dot{\rho}_t^2 + \dot{\phi}_t^2))^2} \tag{4.3}$$

Integration yields:

$$P(\dot{\phi}_t) d\dot{\phi}_t = d\dot{\phi}_t \frac{a_t}{2(a_t + \dot{\phi}_t^2)^{3/2}} \tag{4.4}$$

with $a_t = (C_{tt} C_{tt}^{(1,1)} - (C_{tt}^{(1,0)})^2)/C_{tt}^2 = \partial_t \partial_{t'} \ln |c_{tt'}| |_{t'=t}$. For a stationary process $a_t = a = -c''(0)$. For stationary increments $a_t = C_2(0)/C_{tt} - \frac{1}{4}(\partial_t \ln C_{tt})^2$. Note that this distribution is broad, it does have a first moment but no second moment i.e. $\langle \dot{\phi}_t^2 \rangle$ is infinite.

4.3 Periodized winding

Next one can compute two time correlations of the winding angle. The two time probability measure of the process can be written:

$$\frac{r_t r_{t'} dr_t dr_{t'} d\phi_t d\phi_{t'}}{(2\pi)^2 \Delta_{tt'}} \exp(-\frac{C_{t't'} r_t^2 + C_{tt} r_{t'}^2 - 2C_{tt'} r_t r_{t'} \cos(\phi_t - \phi_{t'})}{2\Delta_{tt'}}) \tag{4.5}$$

with $\Delta_{tt'} = C_{tt} C_{t't'} - C_{tt'}^2$ hence integration over r_t and $r_{t'}$ allows to obtain the probability distribution of $\cos(\phi_t - \phi_{t'})$. Equivalently this gives the probability of $\phi := \phi_t - \phi_{t'}$ modulo 2π, i.e it gives the periodized probability $\tilde{P}(\phi) = \sum_{m=-\infty}^{+\infty} P(\phi + 2\pi m)$ where $P(\phi)$ is the probability of the total winding $\phi \in]-\infty, +\infty[$. The probability distribution allows to compute the correlation functions $\mathcal{C}_n(t,t') = \langle e^{in(\phi_t-\phi_{t'})}\rangle$ for any *integer* n, for instance closed expressions for $\mathcal{C}_n(t,t') = F_n(c_{tt'})$ as a function of the matrix $C_{tt'}$:

$$\begin{aligned} F_1(c) &= \frac{1}{c}(E(c^2) + (c^2-1)K(c^2)) \\ F_2(c) &= 1 + (\frac{1}{c^2} - 1)\ln(1-c^2). \end{aligned} \tag{4.6}$$

with E, K the respective elliptic integral functions. We have checked these results numerically for several stationary processes where $c_{tt'} = c(\tau) = C(\tau)/C(0)$ where $\tau = t - t'$. The process ξ_t^i was generated numerically using a discrete Fourier transform of $\sqrt{\tilde{c}(\omega)N\Delta\tau}\mathcal{A}^i$, where N the number of points is typically $N = 2^{16}$, $\Delta\tau = .01$ is the time segment in the process and $\mathcal{A}^i$ is a unit white gaussian process. We computed $\mathcal{C}_n(\tau)$ where the average $\langle e^{in\phi}\rangle$ is over the time range and over several realizations, typically 10. We plotted $\mathcal{C}_n(\tau)$ parametrically as a function of $c(\tau)$ for various type of noises. Up to numerical accuracy all the curves fall on the predicted master curve $\mathcal{C}_n(\tau) = F_n(c(\tau))$. When $c(\tau)$ is non monotonous, the master curve may be traced more than once. This is illustrated in the left panel of figure 7.

4.4 Variance of the total winding angle

The previous results are easy to derive, and are simple functions of $c_{tt'}$, but they do not contain information about integer winding. They only probe $\tilde{P}(\phi)$, the periodized winding angle distribution. An interesting question is how to access the full winding distribution $P(\phi)$ and whether its dependence on the matrix $C_{tt'}$ remains tractable. It is a more difficult question since to compute the full winding angle one must follow somehow the time evolution of the process, e.g. use that $\phi = \phi_t - \phi_{t'} = \int_{t'}^{t} d\phi_s$. A related difficult question, which requires the full distribution $P(\phi)$, is to obtain the averages $\mathcal{C}_n(t,t') = \langle e^{in(\phi_t-\phi_{t'})}\rangle$ for *non integer* n. It is seen on the right panel of figure 7 that these are not simple functions, but rather unknown and more complicated functionals, of $c_{tt'}$.

We present the simplest result for this question, the variance of the winding angle. Here,

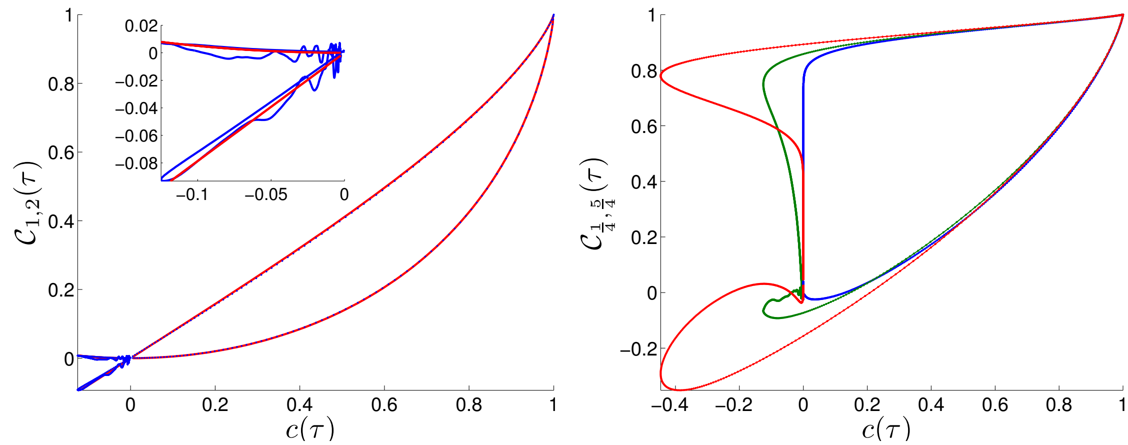

Figure 7: **Left panel**: The correlation function $\mathcal{C}_n(\tau)$ for $n = 1$ (top curve) and for $n = 2$ (bottom curve) as a function of $c(\tau)$ for $c(\tau) = \frac{1-\tau^2}{(1+\tau^2)^2}$ (Blue curves) and the prediction for $F_1(c)$ and $F_2(c)$ (Red curves). The inset shows how the curve is retraced for negative values of c. **Right panel**: The correlation function $\mathcal{C}_n(\tau)$ for $n = \frac{1}{4}$ (three curves starting on the top) $\frac{5}{4}$ (three curves starting on the bottom) as a function of $c(\tau)$ for three processes: $c(\tau) = \exp -\tau^2/2$ (in blue), the $c(\tau)$ used in left panel (in green), $c(\tau) = (1-\tau^2)\exp -\tau^2/2$ (in red). Note that for each n the three curves remain very close for $c > 0.4$ and that for $n = 5/4$ all processes change sign.

for simplicity, and to avoid stochastic calculus subtleties, we restrict to a smooth, i.e. differentiable process. We compute the two time angular velocity correlations

$$\mathcal{C}_v(t,t') = \langle \dot{\phi}_t \dot{\phi}_{t'} \rangle = \frac{1}{2}(\frac{-C_{tt'}^{(1,1)} C_{tt'} + C_{tt'}^{(1,0)} C_{tt'}^{(0,1)}}{C_{tt'}^2}) \ln(1 - c_{tt'}^2)$$
$$= -\frac{1}{2}(\partial_t \partial_{t'} \ln |c_{tt'}|) \ln(1 - c_{tt'}^2) \tag{4.7}$$

where we recall that $c_{tt'} = C_{tt'}/\sqrt{C_{tt} C_{t't'}}$. The variance of the winding angle is then obtained

$$\Phi_{tt'} = \langle (\phi_t - \phi_{t'})^2 \rangle = \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 \mathcal{C}_v(t_1, t_2) \tag{4.8}$$

For stationary processes $c_{tt'} = c(\tau) = C(\tau)/C(0)$ with $\tau = t - t'$. The angular velocity correlation becomes $\mathcal{C}_v(t,t') = \mathcal{C}_v(t-t')$ with:

$$\begin{aligned} \mathcal{C}_v(\tau) &= \frac{1}{2}\left(\frac{C''(\tau)C(\tau) - C'(\tau)^2}{C(\tau)^2}\right)\ln(1-c(\tau)^2) \\ &= \frac{1}{2}(\partial_\tau^2 \ln|c(\tau)|)\ln(1-c(\tau)^2) \end{aligned} \tag{4.9}$$

And the winding angle become $\Phi_{tt'} = \Phi(t-t')$

$$\partial_\tau \Phi(\tau) = 2\int_0^\tau ds \frac{c'(s)^2}{1-c(s)^2} + \frac{c'(\tau)}{c(\tau)}\ln(1-c(\tau)^2) \tag{4.10}$$

For processes such that $c(+\infty) = 0$ we find that the generic behavior is that the winding angle diffuses at large time as $\Phi(\tau) \sim 2D\tau$ with a diffusion coefficient:

$$D = \int_0^\infty ds \frac{c'(s)^2}{1-c(s)^2} \tag{4.11}$$

an integral which converges at small s values when the process is smooth since $c'(0) = 0$.

Examples of some of the non-generic situations where winding angle diffusion does not occur is $c(\tau) = J_0(\tau)$ for which (4.11) is log-divergent at large s and one finds superdiffusion $\Phi(\tau) \sim \frac{2}{\pi}\tau \ln \tau$ at large τ. The above predictions are checked numerically in Fig. 8 in the time variable τ, and as a parametric plot using $c(\tau)$ in Fig. 9, for the diffusive and superdiffusive case.

We now study non-stationary processes, such processes often occur in the context of aging or coarsening dynamics [59, 60]. In some cases these processes can be mapped onto a stationary process using the property of reparametrization of time: if the process $c_{tt'}$ has a winding angle ϕ_t then the process $c_{g(t)g(t')}$ has a winding angle $\phi_{g(t)}$ for any positive monotonic function $g(t)$. Hence for processes of the form $c_{tt'} = \hat{c}(g(t)-g(t'))$, the variance of the winding angle is obtained as $\Phi_{tt'} = \hat{\Phi}(g(t)-g(t'))$ where $\hat{\Phi}(\tau)$ is the variance for the stationary process $\hat{c}(\tau)$. Hence diffusion in $\hat{\Phi}(\tau) \sim 2D\tau$ implies $\Phi_{tt'} = 2(g(t)-g(t'))D$. Among non-stationary processes, processes with stationary increments are of special importance. One such process is the fractional Brownian motion (FBM), $C_{tt'} = \frac{1}{2}(t^{2h} + (t')^{2h} - |t-t'|^{2h})$, with $0 < h < 1$.

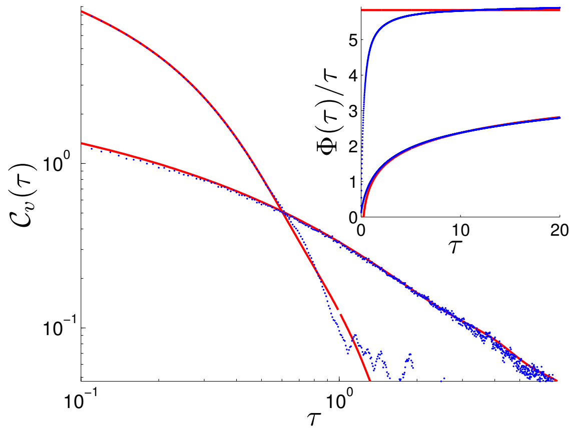

Figure 8: The angular velocity correlation function $\mathcal{C}_v(\tau)$ as a function of τ for the $c(\tau)$ used in Fig.1 (the blue curve with the stronger decay), and for $c(\tau) = J_0(\tau)$ (the second blue curve) together with the predictions of Eq. (4.9) (red curves). In the inset the winding angle variance $\Phi(\tau)$, divided by τ, is displayed for the same two cases. From the top, the first function (in blue) saturates to its diffusive value (red line), with $D \sim 2.92$ calculated from Eq. (4.11). The second function (in blue) is compared with the superdiffusion prediction $\Phi(\tau) = \frac{2}{\pi}\tau \log\tau + 0.907\tau$ from Eq.(4.10). Both results are an average over 50 realizations

For $h = 1/2$ one recovers the standard Brownian motion (BM). The FBM with $h > 1/2$ is smooth and the above mapping applies with the time change $g(t) = \ln t$

$$\hat{c}(s) = \cosh(hs) - 2^{2h-1}|\sinh(s/2)|^{2h} \tag{4.12}$$

can be used, leading to diffusion for the winding angle in the variable $g(t) = \ln t$ at large times, i.e. $\Phi_{tt'} \sim 2D_h \ln(t/t')$ where $D_h = \int_0^\infty \hat{c}'(s)^2/(1-\hat{c}(s)^2)$ diverges as $h \to 1/2^+$. The cases of the Brownian motion $h = 1/2$ require a different expression since we can only address smooth processes, a general smooth process with stationary increments is

$$C_{tt'} = \frac{1}{2}(f(t) + f(t') - f(t-t')). \tag{4.13}$$

The choice $f(t) \sim t$ at large t corresponds to the BM. One possible choice for a smooth process is $f(t) = t - 1 + e^{-t} \sim t^2/2$ at short times. Taking the large t limit at fixed $\tau = t - t_2$

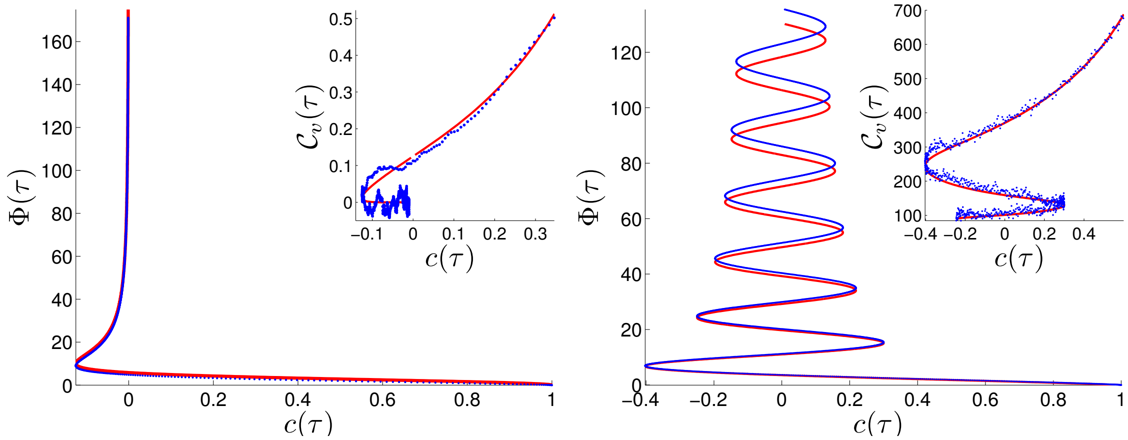

Figure 9: **Left panel:** Parametric plot of the winding angle variance $\Phi(\tau)$ (y-axis) and $c(\tau)$ (x-axis) for the $c(\tau)$ used in Fig.1 (blue curves) and the linear diffusion formula $\Phi(\tau) = 2D\tau$ with $D \sim 2.92$ as predicted from Eq. (4.11) (in red). The inset shows the angular velocity correlation function $\mathcal{C}_v(\tau)$ as a function of $c(\tau)$ for for the same choice of $c(\tau)$ (blue curves) and the results of Eq. (4.9) (red curves). **Right panel:** Parametric plot of the winding angle variance $\Phi(\tau)$ as a function of $c(\tau)$ for $c(\tau) = J_0(\tau)$ (blue curves) and the asymptotic prediction $\Phi(\tau) = \frac{2}{\pi}\tau \log \tau + 0.907\tau$ calculated from Eq. (19) (in red). In the inset the correlation function $\mathcal{C}_v(\tau)$ is shown as a function of $c(\tau)$ for the same $c(\tau)$ (blue curves), together with the results of Eq. (4.9) (red curves). Both results are an average over 50 realizations

we find

$$\Phi_{tt'} \sim \frac{1}{2}(\ln t)^2 \tag{4.14}$$

and recover known behavior of the BM [61]. This was verified numerically in Fig. 10.

4.5 Algebraic area enclosed

Finally we can study the algebraic area A_t enclosed by the process, which satisfies $\dot{A}_t = \frac{1}{2}(\xi_t^x \dot{\xi}_t^y - \xi_t^y \dot{\xi}_t^x)$. Its variance is $\mathcal{C}_A(t,t') = \langle \dot{A}_t \dot{A}_{t'} \rangle$. For a smooth stationary process one

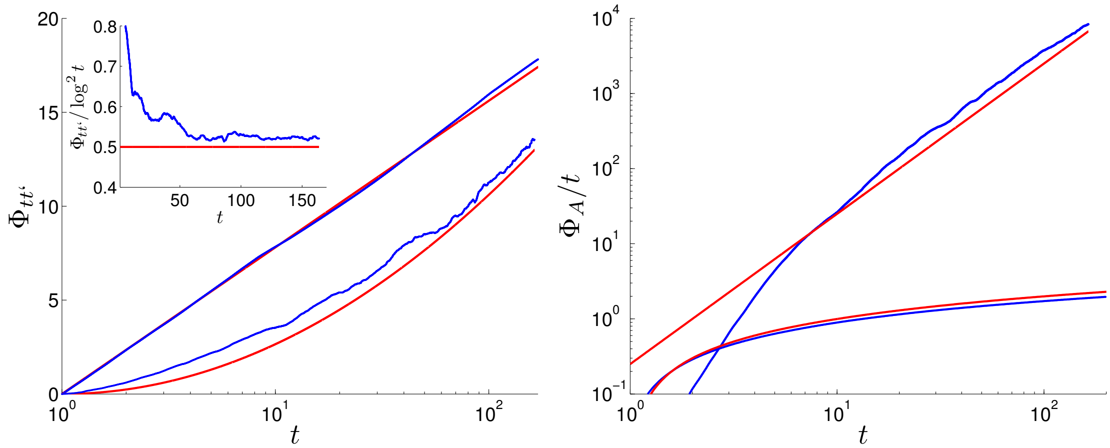

Figure 10: **Left panel** The variance of the winding angle $\Phi_{tt'}$ for $t' = 1$ as a function of t for the random walk, i.e. $C_{tt'}$ of Eq. (4.13) with $f(t) = t - 1 + e^{-t}$ (bottom blue curve) as compared to the asymptotic prediction of Eq.(4.14) (corresponding red curve). The correlation functions $\Phi_{tt'}$ for $t' = 1$ as a function of t for the FBM with $h = 0.6$ calculated using the equivalent stationary process (4.12) with the time reparametrization $s \to e^t$ (top blue curve). The asymptotic diffusion prediction, $\Phi_{tt'} = 2D_{h=0.6} \log t$ where Eq. (4.11) gives $D_{h=0.6} \approx 1.7$ is also shown (top red curve). The results for the random walk required an average over $\sim 10^3$ realizations. **Inset**:The ratio $\Phi_{tt'}/\ln^2 t$ for the random walk, same data, showing the convergence towards 1/2 as predicted in (4.14). **Right panel**: The variance of the algebraic area $\Phi_A = \langle [A_t - A_{t'}]^2 \rangle$ with $t' = 1$ as defined above for : (i) the random walk, i.e. $C_{tt'}$ as in Fig. 8 (blue curve on top at large time) and the asymptotic prediction $\sim t^2/4$ (corresponding red curve) (ii) the result for the stationary process $c(\tau)$ of the left panel of figure 7 as a function of $t \equiv \tau$ (bottom curve) and the asymptotic prediction $2D_A t$ with $D_A = 3\pi/8$ (corresponding red curve). In the inset the ratio Φ_A/t^2 is plotted for the random walk as a function of t, and shows saturation towards the predicted prefactor 1/4.

finds $\mathcal{C}_A(\tau) = -\frac{1}{2}C''(\tau)C(\tau) + \frac{1}{2}C'(\tau)^2$, and the diffusion result $\langle A^2(\tau)\rangle \sim 2D_A\tau$ with $D_A = \int_0^\infty C'(s)^2 ds$. For the random walk $f(t) \sim t$ we find $\langle A_t^2 \rangle \sim t^2/4$ at large t. This is larger than the result for Brownian paths constrained to come back to their starting points (loops) obtained in Ref. [62, 63]. This is well confirmed by our numerics displayed in Fig. 10, where the result for a stationary process which exhibits only diffusive growth of the area is also shown.

4.6 Conclusion

We have computed the angular velocity correlation of a very general smooth Gaussian process in the plane. This allowed us to obtain a simple closed formula for the diffusion coefficient of the winding angle valid for most such stationary processes. Our formula also extends to non-stationary processes, and we derive an expression for the algebraic area enclosed by such processes.

For the Langevin equation of section 2.5 with $m, \eta \to 0$ we find a diffusive fluctuation $\Phi(\tau) = 2D\tau$ with $D \sim 2.92$, in contrast to the large η form of 2.31. The main relevance of this section is, however, to the theory of classical random processes.

5 Summary

In the first part of this work we calculated the renormalized dissipation parameter of a particle in a ring in presence of a dissipative Caldeira-Legget (CL) environment and an external force. Using a renormalization group reasoning we found that the renormalized dissipation η^R flows to a fixed point $\eta_c = \hbar/2\pi$ for large $\eta > \eta_c$. We also studied the Langevin equation which describes the semiclassical limit of this model, and expanded the model for a dirty metal type of environment.

The flow to a fixed point is related to a known quantization of the relaxation resistance in a Coulomb box with a single channel. The mapping between the models assumes many channels and we found that a certain average which contains the relaxation resistance is quantized for large η.

We considered the condition for a proposed box experiment. The field E should be sufficiently small so that g_R is sufficiently near the fixed point. For an initial $g \approx 1$ integration of $\partial g_R/\partial \ln E = g_R^2$ yields $g_R = 1/\ln(\hbar\omega_c/E) \ll g$. E.g. for $g_R \lesssim 0.1$ and a typical $\hbar\omega_c \approx 1$ meV one needs $E/\hbar \lesssim 10^8$Hz. $E/\hbar$ has frequency units, corresponding to 10^8 electrons/sec flowing into the box. We propose measuring the charge fluctuations (noise) $S_Q(\omega) = e^2\langle \hat{N}_t \hat{N}_{t'}\rangle_\omega$ at a frequency, temperature and level spacings Δ such that $\Delta < \omega, T \ll 10^8$Hz, to yield the DC response (2.7) and (2.78). We predict then that the noise $S_Q(\omega)(\frac{2E_c}{e\hbar})^2\frac{1}{\omega} = \frac{\hbar}{\eta^R} = 2\pi$ is quantized.

In the second part of the work we studied the equilibrium properties of the particle, using perturbation in either large or small values of η and using a Monte-Carlo (MC) algorithm. For the algorithm we found that a sign problem emerges and prevents identification of the η^R and hence numerical verification of the fixed point was not feasible.

In the third part of the work we studied a problem in the theory of classical random processes. Given a general two dimensional Gaussian processes on a plane we asked what are the properties of the angle ϕ_t around the center. This problem has some relevance to the semi-classical limit of our model if the process corresponds to a CL environment. We found that for stationary processes with correlation $c(s)$ the angle diffuses with a diffusion coefficient $D = \int_0^\infty ds \frac{c'(s)^2}{1-c(s)^2}$ if this integral is finite. For a random walk process the variance of the angle grows as $\frac{1}{2}\ln^2 t$, and the variance of the algebraic area enclosed by a stationary process diffuses as $D_A = \int_0^\infty c'(s)ds$.

Appendix A Derivation of the semiclassical Retarded Green's function

A.1 Detailed derivation of the 1st order term

First order perturbation of the Green's function

$$R^{(1)}_{t,t'} = -i\frac{1}{2}\int_{t_1,t_2} B_{t_1,t_2}\left\langle \hat{\theta}_{t_1}\hat{\theta}_{t_2}\cos(\theta_{t_1}-\theta_{t_2})\hat{\theta}_{t'}\theta_t\right\rangle_{S_0} = \tag{A.1}$$
$$\frac{-i}{4}\int_{t_1,t_2} B_{t_1,t_2}\sum_{\sigma=\pm}\partial_{\alpha_{i=1,2,3,4}}\ \mathrm{Exp}\Big[i\alpha_1\hat{\theta}_{t_1}+i\alpha_2\hat{\theta}_{t_2}+i\sigma\theta_{t_1}-i\sigma\theta_{t_2}+i\alpha_3\hat{\theta}_{t'}+i\alpha_4\theta_t\Big]|_{\alpha_i=0}$$

An Averaging with Gaussian weight

$$\left\langle \mathrm{e}^{i\theta_{t_1}+i\theta_{t_2}+\ldots+i\hat{\theta}_{t_1}+i\hat{\theta}_{t_2}+\ldots}\right\rangle = \mathrm{e}^{i\left\langle\theta_{t_1}+i\theta_{t_2}+\ldots\right\rangle}\ \mathrm{e}^{-\left\langle(\theta_{t_1}+\theta_{t_2}+\ldots)(\hat{\theta}_{t_1}+\hat{\theta}_{t_2}+\ldots)\right\rangle} =$$
$$\mathrm{e}^{ivt_1+ivt_2+\ldots}\mathrm{e}^{iR_{t_1,t_2}+iR_{t_2,t_1}+\ldots}. \tag{A.2}$$

The retarded function

$$R^{(1)}_{t,t'} =$$
$$\frac{1}{4i}\int_{t_1,t_2}\sum_{\sigma=\pm}\partial_{\alpha_i}B_{t_1,t_2}\ \mathrm{e}^{i\alpha_1(-\sigma R_{t_2,t_1}+\alpha_4 R_{t,t_1})+i\alpha_2(\sigma R_{t_1,t_2}-+\alpha_4 R_{t,t_1})+i\alpha_3(\sigma R_{t_1,t'}-\sigma R_{t_2,t'}+\alpha_4 R_{t,t_1})}\mathrm{e}^{i\sigma v(t_1-t_2)} =$$
$$\frac{1}{4}\int_{t_1,t_2}\sum_{\sigma=\pm}\partial_{\alpha_4}B_{t_1,t_2}(\sigma R_{t_2,t_1}-\alpha_4 R_{t,t_1})(\sigma R_{t_1,t_2}+\alpha_4 R_{t,t_1})(\sigma R_{t_1,t'}-\sigma R_{t_2,t'}+\alpha_4 R_{t,t_1})\mathrm{e}^{i\sigma v(t_1-t_2)} =$$
$$-\int_{t_1,t_2} B_{t_1,t_2}\cos v(t_1-t_2)R_{t,t_1}R_{t_1,t_2}(R_{t_1,t'}-R_{t_2,t'}) \tag{A.3}$$

In the last expression we use $R_t R_{-t} = 0$.

A.2 Derivation of the 2nd order term

Using the same procedure for the second order

$$R^{(2)}_{t,t'} = \frac{i}{2}\left\langle \hat{\theta}_{t'}\theta_t(S_{int})^2\right\rangle = \tag{A.4}$$
$$-\frac{i}{8}\int_{t_1,t_2,t_3,t_4} B_{t_1,t_2}B_{t_3,t_4}\left\langle \hat{\theta}_{t_1}\hat{\theta}_{t_2}\cos(\theta_{t_1}-\theta_{t_2})\hat{\theta}_{t_3}\hat{\theta}_{t_4}\cos(\theta_{t_3}-\theta_{t_4})\hat{\theta}_{t'}\theta_t\right\rangle = \frac{1}{2^5 i}\int_{t_{1..4}} B_{t_1,t_2}B_{t_3,t_4}$$
$$\times\sum_{\sigma_1,\sigma_2=\pm}\partial_{\alpha_{i=1..6}}\left\langle \mathrm{e}^{i\alpha_1\hat{\theta}_{t_1}+i\alpha_2\hat{\theta}_{t_2}+i\alpha_3\hat{\theta}_{t_3}+i\alpha_4\hat{\theta}_{t_4}+i\sigma_1\theta_{t_1}-i\sigma_1\theta_{t_2}+i\sigma_2\theta_{t_3}-i\sigma_2\theta_{t_4}+i\alpha_3\hat{\theta}_{t'}+i\alpha_4\theta_t}\right\rangle|_{\alpha_i=0}$$

using the symmetry between $\sigma_1 \leftrightarrow -\sigma_1$ and $t_1 \leftrightarrow t_2$ and similarly for t_3, t_4

$$R^{(2)}_{t,t'} = \frac{1}{8}\int_{t_1,t_2,t_3,t_4} B_{t_1,t_2}B_{t_3,t_4}\mathrm{e}^{iv(t_1-t_2)-iv(t_3-t_4)}\partial_{\alpha_6}\left[-R_{t_2,t_1}+R_{t_3,t_1}-R_{t_4,t_1}+\alpha_6 R_{t,t_1}\right]$$
$$\left[R_{t_1,t_2}+R_{t_3,t_2}-R_{t_4,t_2}+\alpha_6 R_{t,t_2}\right]\left[R_{t_1,t_3}-R_{t_2,t_3}-R_{t_4,t_3}+\alpha_6 R_{t,t_3}\right]$$
$$\left[R_{t_1,t_4}-R_{t_2,t_4}+R_{t_3,t_4}+\alpha_6 R_{t,t_4}\right]\left[R_{t_1,t'}-R_{t_2,t'}+R_{t_3,t'}-R_{t_4,t'}+\alpha_6 R_{t,t'}\right] \tag{A.5}$$

the choice $t_1 > t_2, t_3, t_4$, only R_{t,t_1} remains. R_τ is real, we separate the exponent to two sinus and two cosine terms as follow

$$R^{(2)}_{t,t'} =$$
$$\frac{1}{8}\int_{t_1,t_2,t_3,t_4} B_{t_1,t_2}B_{t_3,t_4}\left(\cos v(t_1-t_2)\cos v(t_3-t_4)-\sin v(t_1-t_2)\sin v(t_3-t_4)\right)R_{t,t_1}$$
$$\left[R_{t_1,t_2}+R_{t_3,t_2}-R_{t_4,t_2}\right]\left[R_{t_1,t_3}-R_{t_2,t_3}-R_{t_4,t_3}\right]\left[R_{t_1,t_4}-R_{t_2,t_4}+R_{t_3,t_4}\right]$$
$$\left[R_{t_1,t'}-R_{t_2,t'}+R_{t_3,t'}-R_{t_4,t'}\right] \tag{A.6}$$

This long multiplicity of R_t terms is now separated to 8 different terms. Four of these terms are symmetric in $t_3 \leftrightarrow t_4$, and four are antisymmetric. One of the symmetric terms will vanish, we calculate explicitly the other 3 terms, which we label by a to c. Term 'a':

$$R^a_{t,t'} = \frac{1}{2}\int_{t_{1..4}} B_{t_1,t_2}\cos v(t_1-t_2)\times$$
$$R_{t,t_1}R_{t_1,t_2}(R_{t_1,t'}-R_{t_2,t'})B_{t_3,t_4}\cos v(t_3-t_4)(R_{t_1,t_3}-R_{t_2,t_3})(R_{t_1,t_4}-R_{t_2,t_4}) =$$
$$\frac{1}{2}\int_{t_1,t_2} B_{t_1,t_2}\cos v(t_1-t_2)R_{t,t_1}R_{t_1,t_2}(R_{t_1,t'}-R_{t_2,t'})\ \tilde{C}_{t_1,t_2} \tag{A.7}$$

This term in ω space

$$R^a_\omega = -\frac{1}{2}R^2_\omega \int_t R_t B_t \cos vt\ (\mathrm{e}^{i\omega t} - 1)\ \tilde{C}_t$$

with $\tilde{C}_t = 2(C^{(1)}_{t=0} - C^{(1)}_t)$. Similarly we choose two different terms 'b' and 'c' and write them directly in ω space

$$R^b_\omega = R^2_\omega \int_t R^{(1)}_t B_t \cos vt\ (\mathrm{e}^{i\omega t} - 1) \tag{A.8}$$

$$R^c_\omega = R^3_\omega \left[\int_t R_t B_t \cos vt\ (\mathrm{e}^{i\omega t} - 1)\right]^2 = R^{-1}_\omega (R^{(1)}_\omega)^2 \tag{A.9}$$

note the $R^{(1)}_t$ in the expression R^b is the first order result of the retarded green function. R^c_ω is the reducible term containing multiplication of $R^{(1)}_\omega$. Renormalized η for small v is

$$\begin{aligned}
&\frac{1}{\eta^a_2} = \frac{1}{2}\frac{1}{\eta^2}\int_t R_t B_t \tilde{C}(t)\ t = \frac{\hbar}{\pi\eta^3}\int_t R_t B_t\ t\,(\log t + \gamma + \mathcal{O}(v) + \mathcal{O}(1/t)) = \\
&-\frac{\hbar^2}{2\pi^2\eta^3}\log^2 v + \mathcal{O}(v) \\
&\frac{1}{\eta^b_2} = -\frac{\hbar}{\pi\eta^2}\int_t R^{(1)}_t B_t\ t = -\frac{\hbar}{\pi\eta^3}\int_t R_t B_t\ t\,(\log t + \gamma + 1 + \mathcal{O}(v) + \mathcal{O}(1/t)) = \\
&\frac{\hbar^2}{2\pi^2\eta^3}\log^2 v - \frac{\hbar^2}{2\pi^2\eta^3}\log v + \mathcal{O}(v) \\
&\frac{1}{\eta^c_2} = \frac{1}{\eta^3}\left[\int_t R_t B_t\ t\right]^2 = \frac{\hbar^2}{2\pi^2\eta^3}\left[\log v + \mathcal{O}(v)\right]^2 = \frac{\hbar^2}{2\pi^2\eta^3}\log^2 v + \mathcal{O}(v)
\end{aligned} \tag{A.10}$$

We identify four terms antisymmetric in $t_3 \leftrightarrow t_4$ three of those terms will be of $\mathcal{O}(v)$ for small v, and one term which we label d is the following, note that this term is applicable only for the nonequilibrium case. In the Equilibrium case this term vanish (section 2.4.2)

$$\begin{aligned}
R^d_\omega &= -R^2_\omega \int_{t_1,t_2} R_{t_1} R_{t_2} B_{t_1} B_{t_2} \sin vt_1 \sin vt_2 (1 - \mathrm{e}^{i\omega t_1}) \int_{t_3} (R_{t_1+t_3} - R_{t_3}) \\
\frac{1}{\eta^d_2} &= -\frac{1}{\eta^2}\int_{t_1} R_{t_1} B_{t_1} \sin vt_1\ t_1^2 \int_{t_2} R_{t_2} B_{t_2} \sin vt_2 = \frac{\hbar^2}{\pi^2\eta^3}\frac{1}{v} \times v\log v + \mathcal{O}(v) = \\
&\quad \frac{\hbar^2}{\pi^2\eta^3}\log v + \mathcal{O}(v)
\end{aligned} \tag{A.11}$$

We used here

$$\int_{t_3}(R_{t_1+t_3}-R_{t_3})=\frac{1}{\eta}\int_{-t_1}^{0}\left(1-\mathrm{e}^{-(t_1+t_3)\frac{\eta}{m}}\right)+\frac{1}{\eta}\int_0^{\infty}\left(\mathrm{e}^{-t_3\frac{\eta}{m}}-\mathrm{e}^{-(t_1+t_3)\frac{\eta}{m}}\right)=\frac{t_1}{\eta} \quad (A.12)$$

With the term R^d_ω we got $b=-1$, where without it $b=0$.

Appendix B Derivation of the full Retarded function

B.1 Derivation of the S_{int} perturbation

In this appendix we calculate the renormalized dissipation with $R_\tau \to \frac{1}{\eta}\Theta(\tau)\mathrm{e}^{-\delta\tau}$ where $\delta \to 0$ and B_ω from Eq. (2.38). The first order from (2.60) is

$$\begin{aligned}\frac{1}{\eta_1^R} &= \lim_{\omega\to 0} -\frac{2}{\eta^2\hbar}\sin(\frac{\hbar}{2\eta})\int_t B_t\cos vt\ \mathrm{e}^{-\delta t}\ t=\frac{2}{\eta}\sin(\frac{\hbar}{2\eta})\partial_v\int_\omega\frac{|\omega|}{1+\omega^2\tau_0^2}\frac{v}{v^2-\omega^2}=\\ &\frac{2}{\pi\eta}\sin(\frac{\hbar}{2\eta})[\ln(v\tau_0)+1] \qquad (B.1)\end{aligned}$$

For the second order perturbation of the retarded Green's function

$$\begin{aligned}R^{(2)}_{t,t'} &= \frac{i}{2}\left\langle\hat\theta_{t'}\theta_t S^2_{int}\right\rangle_{S_0}=\frac{i}{2^5\hbar^4}\sum_{\epsilon_i,\sigma,\sigma'=\pm}\partial_{\alpha_1,\alpha_2}\int_{t_1,t_2,t_3,t_4}B_{t_1,t_2}B_{t_3,t_4}\epsilon_1\epsilon_2\epsilon_3\epsilon_4\times\\ &\left\langle\mathrm{e}^{i\frac{\hbar}{2}(\alpha_1\hat\theta_{t'}+\alpha_2\theta_t+\epsilon_1\hat\theta_{t_1}+\epsilon_2\hat\theta_{t_2}+\epsilon_3\hat\theta_{t_3}+\epsilon_4\hat\theta_{t_4})+i\sigma(\theta_{t_1}-\theta_{t_2})+i\sigma'(\theta_{t_3}-\theta_{t_4})}\right\rangle|_{\alpha_i=0}=\\ &\frac{-i}{2^4\hbar^4}\sum_{\epsilon_i,\mu=\pm}\int_{t_1,t_2,t_3,t_4}B_{t_i}\epsilon_1\epsilon_2\epsilon_3\epsilon_4A_2\cos[v(t_1-t_2)+\mu v(t_3-t_4)]\times\\ &[iR_{t,t'}-\tfrac{1}{2}\hbar(\epsilon_1R_{t,t_1}+\epsilon_2R_{t,t_2}+\epsilon_3R_{t,t_3}+\epsilon_4R_{t,t_4})(R_{t_1,t'}-R_{t_2,t'}+\mu R_{t_3,t'}-\mu R_{t_4,t'})]\\ A_2 &= \exp\{i\frac{\hbar}{2}\epsilon_1(-R_{t_2,t_1}+\mu R_{t_3,t_1}-\mu R_{t_4,t_1})+i\frac{\hbar}{2}\epsilon_2(R_{t_1,t_2}+\mu R_{t_3,t_2}-\mu R_{t_4,t_2})\}\times\\ &\exp\{i\frac{\hbar}{2}\epsilon_3(R_{t_1,t_3}-R_{t_2,t_3}-\mu R_{t_4,t_3})+i\frac{\hbar}{2}\epsilon_4(R_{t_1,t_4}-R_{t_2,t_4}+\mu R_{t_3,t_4})\} \qquad (B.2)\end{aligned}$$

define $\tau=t_1-t_2$ and then the factor $R_{t_1,t'}-R_{t_2,t'}$ is finite only if either (i) $t'<t_1$ and $t_2<t'$ hence $t'<t_1<t'+\tau$, or (ii) $t_1<t'$ and $t_2>t'$ hence $t'+\tau<t_1<t'$; in both cases and similarly, for $\tau'=t_3-t_4$

$$\int_{t_1}[R_{t_1,t'}-R_{t_2,t'}]\to\frac{1}{\eta}\tau \qquad \int_{t_3}[R_{t_3,t'}-R_{t_4,t'}]\to\frac{1}{\eta}\tau' \qquad t'\to-\infty \qquad (B.3)$$

Therefore appears a factor $[t_1 - t_2 + \mu(t_3 - t_4)] \cos[v(t_1 - t_2) + \mu v(t_3 - t_4)] = \partial_v \sin[v(t_1 - t_2 + \mu v(t_3 - t_4)]$, and since $1/\eta = \lim_{\tau \to \infty} R^R_\tau$

$$\frac{1}{\eta^R_2} = \frac{i}{\eta^2 2^4 \hbar^3} \frac{\partial}{\partial v} \sum_{\epsilon_i, \mu = \pm} \int_{t_2,t_3,t_4} B_{t_1,t_2} B_{t_3,t_4} \epsilon_1\epsilon_2\epsilon_3\epsilon_4 (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4) A_2 \sin[v(t_1 - t_2) + \mu v(t_3 - t_4)]$$

All the $\pm$ index ϵ_i are equivalent in the expression by change of variables $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 \sum_i \epsilon_i = 4\epsilon_2\epsilon_3\epsilon_4$ this means the choice $t_1 > t_{2,3,4}$ and get (2.62). It easy to see

$$\sum_{\epsilon_i, \mu = \pm} \epsilon_2\epsilon_3\epsilon_4 \exp\{i\frac{\hbar}{2}\epsilon_2(R_{t_1,t_2} + \mu R_{t_3,t_2} - \mu R_{t_4,t_2}) + i\frac{\hbar}{2}\epsilon_3(R_{t_1,t_3} - R_{t_2,t_3} - \mu R_{t_4,t_3})\} \times \exp\{i\frac{\hbar}{2}\epsilon_4(R_{t_1,t_4} - R_{t_2,t_4} + \mu R_{t_3,t_4})\} = -8i \sin^2 \frac{\hbar}{2\eta} \sin \frac{\hbar}{\eta} \quad \text{(B.4)}$$

for $t_1 > t_3 > t_2, t_4$ and zero otherwise. The remaining integral is

$$\int_{t1>t3>t2,t4} \sin v(t_1 - t_2 + t_3 - t_4) B_{t_1,t_2} B_{t_3,t_4} = \sum_\sigma \iint_{\omega_1,\omega_2} \frac{B_{\omega_1} B_{\omega_2}}{(\omega_1 + \sigma v)^2 (\omega_2 + \sigma v)} = (2\hbar\eta)^2 v \ln v\tau_0 [\ln v\tau_0 + 1] \quad \text{(B.5)}$$

after v derivation we get the result in (2.63).

Appendix C The published letter

This appendix contains the published letter

- Y. Etzioni, B. Horovitz and P. Le Doussal, “Rings and Boxes in Dissipative Environments”, Phys. Rev. Lett. 106, 166803 (2011)

Rings and Boxes in Dissipative Environments

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(Received 21 January 2011; published 19 April 2011)

We study a particle on a ring in presence of a dissipative Caldeira-Leggett environment and derive its response to a dc field. We find, through a 2-loop renormalization group analysis, that a large dissipation parameter η flows to a fixed point $\eta_R = \eta_c = \hbar/2\pi$. We also reexamine the mapping of this problem to that of the Coulomb box and show that the relaxation resistance, of recent interest, is quantized for large η. For finite $\eta > \eta_c$ we find that a certain average of the relaxation resistance is quantized. We propose a box experiment to measure a quantized noise.

DOI: 10.1103/PhysRevLett.106.166803 PACS numbers: 73.23.Ra, 05.40.−a, 05.60.Gg, 73.23.Hk

Two of the most important mesoscopic structures are rings, for the study of persistent currents, and quantum dots or boxes, for the study of charge quantization. Of particular recent interest is the quantization of the relaxation resistance R_q, defined via an ac capacitance of a single electron box (SEB). Following the prediction of Büttiker, Thomas, and Prêtre [1] that $R_q = h/2e^2$ for a single mode resistor, a quantum mesoscopic RC circuit has been implemented in a two-dimensional electron gas [2] and $R_q = h/2e^2$ has been measured. The theory has been recently extended to include Coulomb blockade effects [3] showing that $R_q = h/2e^2$ is valid for small dots and crosses over to $R_q = h/e^2$ for large dots.

In parallel, recent data has observed Aharonov-Bohm oscillations from single electron states in semiconducting rings [4]. Further theoretical works have considered the effects of dissipative environments on a single particle in a ring [5], in particular, studying the renormalization of the mass M^* and its possible relation to dephasing [5–8].

It is rather remarkable that the ring and box problems are related via the AES mapping [9] where the ring experiences a Caldeira-Leggett (CL) [10] environment. While the exact mapping assumes weak tunneling into the box with many channels, it has been extensively used to describe various tunnel junctions [11], the Coulomb blockade phenomena in SEB and in the single electron transistor (SET) [11–21].

In the present work we address the ring problem by the real time Keldysh method and study it using a 2-loop expansion and renormalization group (RG) reasoning. We find that perturbation theory identifies an unexpected new small parameter $\sin(\frac{\hbar}{2\eta})$ where η is the dissipation parameter on the ring, or the lead-dot coupling in the SEB. We infer that a large η flows to a fixed point $\eta_R = \eta_c$ with $\hbar/2\eta_c = \pi$. An intuitive argument for this quantization is given before the conclusions. In Monte Carlo studies [15,18] of M^*, no sign of a finite coupling fixed point has been detected. Our method evaluates the response to a strictly dc electric field E, equivalent to a magnetic flux through the ring that increases linearly with time, hence a nonequilibrium response. We claim that thermodynamic quantities like M^*, that are flux sensitive, decouple from the response to E, a response that averages over flux values.

In terms of the SEB, our results extend the previous analysis [3] to the case of many channels N_c [22]. We note that for $N_c > 1$ the relaxation resistance for noninteracting electrons becomes $h/(2N_c e^2)$ [1]. We find that for strong coupling, $\eta/\hbar \gtrsim 1$ the relaxation resistance is quantized to e^2/h up to an exponentially small correction $\sim e^{-\pi\eta/\hbar}$. For finite η, but still $\eta > \eta_c$ we find that a certain average of the relaxation resistance is quantized [see Eq. (12) below].

We proceed to reexamine the mapping of the box and ring problems. For the SEB one has the action

$$S = \int_t \Big\{ \sum_{\alpha n} d^\dagger_{\alpha n}(i\hbar\partial_t - \epsilon_\alpha) d_{\alpha n} - E_c(\hat{N} - N_0)^2 \Big\} + S_{\rm lead} + S_{\rm tun}, \quad (1)$$

where $d_{\alpha n}$ are dot electron operators, $n = 1, \ldots, N_c$ labels the channels, $\hat{N} = \sum_{\alpha n} d^\dagger_{\alpha n} d_{\alpha n}$, $E_c = e^2/(2C_g)$ with C_g is the geometric (bare) capacitance, N_0 is proportional to the gate voltage, $S_{\rm lead}$ describes free electrons on the lead and $S_{\rm tun}$ is the tunneling between the lead and the dot. We introduce an auxiliary variable θ_t with an action $E_c \int_t [\hat{N} - N_0 - \hbar\dot{\theta}/2E_c]^2$ and rewrite the total action as

$$S = \int_t \Big\{ \sum_{\alpha n} d^\dagger_{\alpha n}(i\hbar\partial_t - \epsilon_\alpha - \hbar\dot{\theta}_t) d_{\alpha n} + \frac{\hbar^2 \dot{\theta}_t^2}{4E_c} + N_0 \hbar \dot{\theta}_t \Big\} + S_{\rm lead} + S_{\rm tun}. \quad (2)$$

In terms of fermion operators $\tilde{d}_{\alpha n} = e^{i\theta(t)} d_{\alpha n}$, integrating out these fermions and expanding in $S_{\rm tun}$ yields the well known effective action for the SEB [9,11–13,15–20]. Equation (2) shows that the equivalent particle on a ring has a mass $M = \hbar^2/(2E_c)$ (the radius of the ring is chosen as $= 1$) and there is a flux (in unit of the flux quantum)

$\phi_x = -N_0$ through the ring. The tunneling amplitudes squared, weighted by the number N_c of channels, become the dissipation parameter η of the particle. The mapping becomes exact in the large N_c limit at fixed η and for small mean level spacing [23] $\Delta \ll E_c$, a situation that can be realized [22]; the application of this mapping is therefore limited to the temperature range $\Delta < T \ll E_c$. Furthermore, by shifting $\hbar\dot{\theta}_t \to \hbar\dot{\theta}_t + 2E_c(\hat{N}_t - N_0)$ we obtain $\hbar\langle\dot{\theta}_t\rangle = 2E_c[\langle\hat{N}\rangle_{N_0} - N_0]$ and also a relation between response functions

$$\hbar^2\tilde{K}_{t,t'} = -2E_c\hbar\delta(t-t') + 4E_c^2 K_{t,t'}, \tag{3}$$

where $\tilde{K}_{t,t'} = +i\theta(t-t')\langle[\dot{\theta}_t, \dot{\theta}_{t'}]\rangle$ is the response for the ring while $K_{t,t'} = +i\theta(t-t')\langle[\hat{N}_t, \hat{N}_{t'}]\rangle$ is for the SEB.

The SEB response is parameterized as [3] $\frac{e^2}{\hbar}K(\omega) = C_0(1 + i\omega C_0 R_q)$ where C_0 is the effective dc capacitance and R_q is the celebrated relaxation resistance [1]. The corresponding $\tilde{K}_{t,t'}$ is the response to a change in the external flux and is parameterized as

$$\tilde{K}(\omega) = -K_0(\phi_x) + i\omega K_1(\phi_x) + O(\omega^2) \tag{4}$$

and the persistent current from a time independent flux is $\langle\dot{\theta}_t\rangle = \int_0^{\phi_x} K_0(\phi_x')d\phi_x'$. The continuation to imaginary time identifies the curvature of the free energy [5–8], or an effective mass, as $\frac{1}{\hbar}\frac{\partial^2 F}{\partial\phi_x^2} = \hbar/M^*(\phi_x) = K_0(\phi_x)$; e.g., without tunneling $M^* = M$ while for large η the effective mass $M^* \sim e^{\pi\eta/\hbar}$ is exponentially large.

Consider now the system in presence of a (classical) electric field E, of Hamiltonian $\delta\mathcal{H}_{\text{ring}} = -(E + \delta E(t))\theta$ and define the linear response $\delta\langle\theta_t\rangle_E = \int_{t'}\mathcal{R}_{t,t'}\delta E(t')$ to a small perturbation δE. This response is studied below for a dc field. In general its low frequency form is [see (8) below] $\mathcal{R}(\omega) = \frac{-1}{i\omega\eta_R(E)}$ which defines $\eta_R(E)$ as a renormalized dissipation parameter. Since $E = \hbar\dot{\phi}_x$ we expect $\hbar\omega^2\mathcal{R}(\omega) = \tilde{K}(\omega)$, hence the K_0 term in Eq. (4) is not reproduced. To resolve this discrepancy we note that an additional constant flux ϕ_x in the total flux $\phi_x + Et/\hbar$ can be eliminated by redefining the origin of the time t, therefore the persistent current part should be eliminated. More precisely, define $\hbar\phi_x(t) = Et$; the 1st term in (4) $K_0(\phi_x) = K_0(Et/\hbar)$ becomes a periodic function, i.e., an ac response at $\omega_E = 2\pi E/\hbar$. For a dc response at finite E this persistent current response averages to zero, i.e., $\int_0^1 K_0(\phi_x)d\phi_x = 0$. The same reasoning applies to a ϕ_x average on $K_1(\phi_x)$. Hence the dc response to a dc field is given by

$$\lim_{E\to 0}\lim_{\omega\to 0}\frac{\tilde{K}(\omega)}{\omega} = i\int_0^1 K_1(\phi_x)d\phi_x. \tag{5}$$

Therefore, $\hbar/\eta_R = \int_0^1 K_1(\phi_x)d\phi_x$ where we denote $\eta_R \equiv \eta_R(E\to 0)$. The order of limits in (5) signifies that η_R is essentially a nonequilibrium response. The physical picture is that in a dc field the particle rotates around the ring and produces two types of currents. First is the persistent current that oscillates in time as ϕ_x increases and is therefore time averaged to zero; this current is nondissipative. Second, there is a genuine dc response from the $i\omega K_1$ term, which is dissipative.

In terms of the SEB response, using Eq. (3), we obtain the following mapping of ring and box parameters as functions of flux ϕ_x and N_0:

$$\begin{aligned} \frac{M}{M^*(\phi_x)} &= 1 - \frac{C_0(N_0)}{C_g}, \\ \frac{\hbar}{\eta_R} &= \frac{e^2}{\hbar}\int_0^1 \frac{C_0^2(N_0)}{C_g^2}R_q(N_0)dN_0, \end{aligned} \tag{6}$$

and we note also that $\int_0^1 C_0(N_0)dN_0 = C_g$.

At this stage we can already propose an interesting experiment for the SEB. By analogy with $E = \hbar\dot{\phi}_x$ in the ring, we propose measuring the response to a gate voltage that is linear in time $N_0 \sim t$. This leads to a dc current into the Coulomb box whose dissipation is the average in Eq. (6). This average is predicted to be quantized, at least for $\eta > \eta_c$, as discussed below.

We proceed now to study the ring problem. To derive the Keldysh action, we start from the well known action of a particle in a CL environment [10] in two dimensions with a position vector $\mathbf{x}_t^{\pm}$, where $\pm$ correspond to the upper and lower Keldysh contour, $S_K = i\int_{t,t'}\hat{\mathbf{x}}_t R_{t,t'}^{-1}\mathbf{x}_{t'} + \frac{1}{2}\int_{t,t'}\hat{\mathbf{x}}_t B_{t,t'}\hat{\mathbf{x}}_{t'}$ and $\mathbf{x}_t = \frac{1}{2}(\mathbf{x}_t^+ + \mathbf{x}_t^-)$ and $\hat{\mathbf{x}}_t = (\mathbf{x}_t^+ - \mathbf{x}_t^-)/\hbar$. The simplest response function $R(\omega)$, in Fourier transform, and the noise function $B(\omega)$, at zero temperature, are $R(\omega) = [M\omega^2 + i\eta\omega]^{-1}$, $B(\omega) = \hbar\eta|\omega|$. This quadratic problem corresponds to a particle of mass M and a friction η within a Langevin equation $M\ddot{\mathbf{x}}_t + \eta\dot{\mathbf{x}}_t = \boldsymbol{\xi}_t$; each component of $\boldsymbol{\xi}_t = (\xi_t^x, \xi_t^y)$ is random with correlations $B(\omega)$.

We project now the position on a ring, i.e., $\mathbf{x}_t^{\pm} = (\cos\theta_t^{\pm}, \sin\theta_t^{\pm})$, and rewrite the action in terms of classical and quantum angle variables $\theta_t = \frac{1}{2}(\theta_t^+ + \theta_t^-)$ and $\hat{\theta}_t = (\theta_t^+ - \theta_t^-)/\hbar$:

$$\begin{aligned} S_K &= S_0 + S_{\text{int}} + S_c, \\ S_0 &= i\int_{t,t'}\hat{\theta}_t R_{tt'}^{-1}\delta\theta_{t'} = i\int_{t,t'}\hat{\theta}_t R_{tt'}^{-1}\theta_{t'} - iE\int_t \hat{\theta}_t, \\ S_{\text{int}} &= \frac{2}{\hbar^2}\int_{t,t'} B_{t,t'}\sin\left(\frac{\hbar}{2}\hat{\theta}_t\right)\sin\left(\frac{\hbar}{2}\hat{\theta}_{t'}\right)\cos(\theta_{t'} - \theta_t), \\ S_c &= \frac{i\eta}{\hbar}\int_t[\sin(\hbar\hat{\theta}_t)\dot{\theta}_{t^-} - \hbar\hat{\theta}_t\dot{\theta}_{t^-}], \end{aligned} \tag{7}$$

where t^- is infinitesimal below t. A Gaussian term S_0 has been singled out so that a perturbation scheme in powers of S_{int}, S_c can be defined. We have added an external electric field E, hence the particle acquires a velocity $v = \langle\dot{\theta}_t\rangle$ as a function of E. To perform a perturbation theory it is convenient to introduce the bare velocity $v_0 = E/\eta$ and to define $\theta_t = \delta\theta_t + v_0 t$. The derivative of the $v(E)$ characteristics is easily shown to be related to $\eta_R(E)$ via

$$\frac{dv}{dE} = i\left\langle \int_{t'} \dot{\theta}_t \hat{\theta}_{t'} \right\rangle = \lim_{t-t'\to\infty} \mathcal{R}_{t,t'} \equiv \frac{1}{\eta_R(E)}, \tag{8}$$

where $\mathcal{R}_{t,t'} = i\langle \theta_t \hat{\theta}_{t'} \rangle$ is the full response function defined above. We note that the form (7) for S_K has been derived also for the SEB [9,11,12,20,21].

The semiclassical limit of (7), which corresponds to small $\hbar/\eta$, is obtained by linearizing the sine terms, and is equivalent to a Langevin equation (also obtained for the SET [24])

$$M\ddot{\theta}_t + \eta\dot{\theta}_t = \xi_t^x \cos\theta_t + \xi_t^y \sin\theta_t + E \tag{9}$$

which is in fact the 2D Langevin equation projected on the tangent to the ring.

We perform a perturbative expansion of the action with respect to $S_{\rm int}$, S_c to compute $\eta_R(E)$. The perturbative expansion of $\eta_R(E)$ exhibits logarithmic divergences when $E \to 0$. The velocity v_0 thus provides a natural low frequency cutoff for this divergences, and the mass provides a high frequency cutoff at $\omega_c = \eta/M$. The expansion terms can be classified as n loops by looking at the small $\hbar/\eta$ power of each term which is of order $R^{2n-1}B^n/\eta^2 \sim \hbar^n/\eta^{n+1}$. However, we find, due to the periodicity of the action in the angle variables, that the R^{2n-1} factors in front of the logarithmic terms become periodic functions: The result up to two loops and $O(v_0)$ is

$$\frac{1}{\eta_R(E)} = \frac{1}{\eta} - \frac{2}{\pi\eta}\sin\left(\frac{\hbar}{2\eta}\right)\ln[v_0/\omega_c'] + \frac{4}{\pi^2\hbar}\sin^2\left(\frac{\hbar}{2\eta}\right) \times \sin\left(\frac{\hbar}{\eta}\right)\{\ln^2[v_0/\omega_c'] + b_0\ln[v_0/\omega_c']\}, \tag{10}$$

where $b_0 = O(1)$ may weakly depend on η and $\omega_c'/\omega_c = 1 + O(1/\eta^2)$. In the semiclassical limit of large η one can reexpress (10) in terms of the small parameter $\gamma = \frac{\hbar}{\pi\eta}$ and $\gamma_R = \frac{\hbar}{\pi\eta_R(E)}$ and obtain the 2-loop β function as $-E\partial_E\gamma_R = \gamma_R^2 - b_0\gamma_R^3 + O(\gamma_R^4)$ which has the equilibrium form [13,14] if $b_0 = -1$. We show in Fig. 1 our numerical solution for Eq. (9) with a reasonable fit to the 2-loop form with $b_0 = 0$. The full quantum theory (7) including its nonequilibrium limit (5) differs from these descriptions [13,14,21].

We consider now the quantum theory, beyond large η. We note that in (10) $g = \frac{2}{\pi}\sin(\frac{\hbar}{2\eta})$ acts as an unexpected small parameter for the expansion, since all divergences vanish when $g = 0$. It raises the interesting possibility that $g = 0$ be viewed as a RG fixed point. For that we need to find a renormalized coupling which obeys multiplicative RG, the simplest choice being $g_R = \frac{2}{\pi}\sin(\frac{\hbar}{2\eta_R(E)})$. The question is then whether the β-function $\beta = -E\partial_E g_R$ can be written only in terms of g_R. Although the nonperiodic $1/\eta$ factor in (10) appears at first problematic, we propose that resummation from higher loops, which allows for higher order terms $O(\frac{1}{\eta^4})$ changes the 1-loop term in (10) by $\frac{\hbar}{2\eta} \to \sin(\frac{\hbar}{2\eta})$, so that by taking a sine of both sides it yields to order g^3

$$g_R = g \pm g^2\ln(v_0/\omega_c') + g^3[\ln^2(v_0/\omega_c') + b_0\ln(v_0/\omega_c')], \tag{11}$$

where $\pm$ refers to $g = 0$ with $\cos(\frac{\hbar}{2\eta}) = \pm 1$, leading to $\beta(g_R) = \mp g_R^2 - b_0 g_R^3 + O(g_R^4)$.

To further motivate this proposal we consider the response $\bar{R}_{t,t'} = i\frac{2}{\hbar}\langle \theta_t \sin(\frac{\hbar}{2}\hat{\theta}_{t'})\rangle$. Physically, $e^{\pm i(\hbar/2)\hat{\theta}_{t'}}$ corresponds to an electric field pulse $\delta E(t) = \pm\frac{\hbar}{2}\delta(t - t')$ or equivalently a rapid change of flux by $\pm\frac{1}{2}$, therefore $\bar{R}_{t,t'}$ corresponds to the difference in response to these two flux pulses. For $\bar{R}_{t,t'}$ the 1-loop term is fully periodic with $\frac{\hbar}{2\eta} \to \sin(\frac{\hbar}{2\eta})$ in Eq. (10). We note that there are many other operators that have vanishing perturbations at $g = 0$ to 2nd order in $S_{\rm int}$, S_c, e.g., the dissipation term in Eq. (7) $\langle \theta_t \sin(\hbar\hat{\theta}_{t'})\rangle$, or the response to an ac field with frequency $v\,\langle \theta_t \cos\delta\theta_{t'} \sin\frac{\hbar}{2}\hat{\theta}_{t'}\rangle$. We propose then that $g = 0$ are exact zeroes of the perturbation expansion and requiring an RG structure leads then to the result (11).

Equation (10) yields fixed points at $\frac{\hbar}{2\eta_n} = n\pi$ with $n = 1, 2, 3, \ldots$ that are attractive at $\eta > \eta_n$ and repulsive at $\eta < \eta_n$; i.e., the flow of $\eta \neq \eta_n$ is always to smaller η. At these fixed points a Gaussian evaluation yields the correlation $\langle \cos\theta_t \cos\theta_0 \rangle \sim t^{-2n}$. We recall now a theorem for the lattice model [25] where the equilibrium action with mass related cutoff is replaced by an action on a lattice resulting in an XY model with long range interactions. The theorem states [25] that $\langle \cos\theta_t \cos\theta_0 \rangle \sim 1/t^2$; this result was also derived in first order in η [8]. The range $\eta > \eta_1$

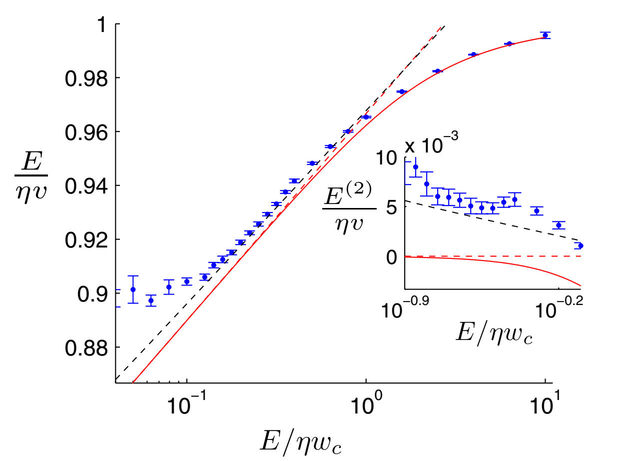

FIG. 1 (color online). Velocity-field relation for Eq. (9) with $\eta = 30\hbar/\pi$. The circles are numerical data, the full line is a 1st order perturbation in $1/\eta$, the dashed lower (red) line is its logarithmic expansion for large $\ln v_0/\omega_c$ ($v_0 = E/\eta$ being the bare velocity) and the dashed upper (black) line includes the 2nd order logarithmic term, corresponding to Eq. (10) for $\hbar \to 0$ and $b_0 = 0$. The 2nd order terms are also shown in the inset after the 1st order is subtracted, i.e., $\frac{E^{(2)}}{\eta v} = \frac{E}{\eta v} - 1 - \frac{\hbar}{\pi\eta}(\ln\frac{v_0}{\omega_c} - 1)$.

has an RG flow to η_1 and is therefore consistent with the theorem. The hypothesis of Gaussian fixed points corresponding to $n \geq 2$ is inconsistent with the theorem, i.e., $\langle \cos\theta_t \cos\theta_0 \rangle$ becomes a relevant operator at the $n \leq 2$ points rendering them unstable. For $\eta < \eta_1$ the system may have non-Gaussian fixed points or a line of fixed points as hinted by the small η perturbation [8]. Note that in the SEB problem $\cos\theta_t$ corresponds to a lead-dot voltage and its correlations determine the SET conductance [9,11,19], while in the ring problem it corresponds to fluctuations in the circular asymmetry.

The special value $\eta_R = \hbar/(2\pi)$ has a topological interpretation as a Thouless charge pump [26]. Consider a slow change of ϕ_x by one unit with $\hbar\dot{\phi}_x = \eta_R \langle \dot{\theta} \rangle$. For $\eta_R = \hbar/(2\pi)$ the total change in the position of the particle is $\int_t \langle \dot{\theta} \rangle dt = 2\pi$, i.e., the particle comes back to the same position on the ring and a unit charge has been transported. Such quantization requires a gap [26], though gapless cases are also known [27,28]. The quantized η_R also results from arguing that there should be a unique frequency $\omega_E = v$ as $E \to 0$, as suggested by linear response.

We conclude that for $\eta > \eta_1 \equiv \eta_R$ the SEB satisfies the quantization

$$\int_0^1 \frac{C_0^2(N_0)}{C_g^2} R_q(N_0) dN_0 = \frac{h}{e^2}. \tag{12}$$

In particular, when $\eta/\hbar \gtrsim 1$ we have from the known $M^*/M \sim e^{\pi\eta/\hbar}$ [5–8] and from Eq. (6) that $C_0/C_g = 1 + O(e^{-\pi\eta/\hbar})$. We expect R_q to be independent of N_0 at large η, hence $R_q = \frac{h}{e^2}[1 + O(e^{-\pi\eta/\hbar})]$, similar to the $N_c = 1$ case [3].

The conductance for the ring can be defined by the voltage around the ring $2\pi E/e$ and the current $e\langle \dot{\theta} \rangle / 2\pi$, hence we expect the conductance for $\eta > \eta_R$ to be

$$G_{\rm ring} = \frac{e^2}{4\pi^2 \eta_R} = \frac{e^2}{h}. \tag{13}$$

Finally, we reconsider the conditions for our proposed box experiment. The field E should be sufficiently small so that g_R is sufficiently near the fixed point. For an initial $g \approx 1$ integration of $\partial g_R / \partial \ln E = g_R^2$ yields $g_R = 1/\ln(\hbar\omega_c/E) \ll g$. For example, for $g_R \lesssim 0.1$ and a typical $\hbar\omega_c \approx 1$ meV one needs $E/\hbar \lesssim 10^8$ Hz. $E/\hbar$ has frequency units, corresponding to 10^8 electrons/sec flowing into the box. We propose measuring the charge fluctuations (noise) $S_Q(\omega) = e^2 \langle \hat{N}_t \hat{N}_{t'} \rangle_\omega$ at a frequency, temperature and level spacings Δ such that $\Delta < \omega$, $T \ll 10^8$ Hz, to yield the dc response (5) and (12). We predict then that the noise $S_Q(\omega)(\frac{2E_c}{e\hbar})^2 \frac{1}{\omega} = \frac{\hbar}{\eta_R} = 2\pi$ is quantized.

Acknowledgements: We thank I. S. Burmistrov, G. Fève, A. Golub, D. Goldhaber-Gordon, S. L. Lukyanov, Y. Meir, C. Mora, B. Plaçais, and G. Zaránd for stimulating discussions. BH acknowledges kind hospitality and financial support from LPTENS and PLD from Ben Gurion University. This research was supported by The Israel Science Foundation (Grant No. 1078/07) and by the ANR grant 09-BLAN-0097-01/2.

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דיסיפציה ודיפוזיה של חלקיקים בסביבה אקראית

מחקר לשם מילוי חלקי של הדרישות לקבלת תואר
"דוקטור לפילוסופיה"

מוגש מאת: **יואב עציוני**

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30 לאוקטובר 2011

באר שבע

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אישור המנחה ________________________.

אישור דיקן בית הספר ללימודי מחקר מתקדמים ע"ש קרייטמן ________________________.

30 לאוקטובר 2011

באר שבע

העבודה נעשתה בהדרכת

פרופ' ברוך הורוביץ

המחלקה לפיסיקה

הפקולטה למדעי הטבע

תקציר

בעבודה זו אנו חוקרים את הדינמיקה של חלקיק על טבעת בנוכחות סביבה דיסיפטיבית מסוג Caldeira-Leggett. בחלק הראשון של העבודה אנו חוקרים מודל זה באמצעות הפורמליזם של Keldysh לדינמיקה מחוץ לשיווי משקל על מנת לקבל את תגובת החלקיק לשדה DC חיצוני. באמצעות תגובת החלקיק אנו מחשבים את פרמטר הדיסיפציה המנורמל η^R עד לסדר שני.על ידי ניתוח של חבורת הרה-נורמליזציה אנו מוצאים כי עבור פרמטר דיסיפציה גדול $\eta > \eta_c$ הפרמטר המנורמל זורם לנקודת שבת ב $\eta_c = \hbar / 2\pi$. אנו חוקרים את הגבול הסמי-קלאסי של הבעיה בו אנו מראים כי את מודל החלקיק אפשר לבטא באמצעות משוואת Langevin אותה אנו גם פותרים נומרית. עבור הגבול הסמי-קלאסי אנו מרחיבים את המודל לסביבה כללית יותר סביבה הנוצרת על ידי מתכת מזוהמת.

אנו בוחנים מחדש מיפוי ידוע של מודל החלקיק למודל של קופסא קולומבית ומוצאים כי ערך תוחלת מסוים של ההתנגדות הוא בדיד עבור ערכי η גדולים ומציעים ניסוי המסוגל למדוד את הגודל הבדיד של הרעש.

בחלק השני של עבודה זו אנו בוחנים את המודל כאשר הוא נמצא בשיווי משקל באמצעות הפורמליזם של Matsubara. אנו מנתחים את המודל הפרעתית סביב ערכי η גדולים וערכי η קטנים. אנו מפתחים אלגוריתם מונטה קרלו על מנת לפתור את הבעיה. עם זאת, כאשר השטף דרך הטבעת הוא חצי, ביחידות של שטף קוונטי, אנו נתקלים בבעיית הסימן הידועה לשמצה, כתוצאה ממנה אין אפשרות לזהות את η^R באמצעות המידע הנומרי.

בחלק האחרון של העבודה אנו חוקרים את התנהגותה של זווית הסיבוב ϕ_t סביב המרכז של רעש גאוסי מישורי בעל פונקציית קורלציה שרירותית, כאשר הקורלציה של רעש מסוג Caldeira-Leggett הוא מקרה פרטי. המוטיבציה לעבודה זו נבעה מהתנהגות החלקיק בגבול הסמי-קלאסי עבור ערכי η קטנים. אנו מקבלים את ההתפלגות הסטציונרית של $\dot{\phi}_t$, נוסחה סגורה כפונקציה של פונקציית הקורלציה עבור השונות של הזווית ϕ_t, את הקורלציה עבור $e^{in\phi_t}$ עבור n שלם ואת השונות של השטח התחום על ידי התהליך. תוצאות אלו נבדקות נומרית.