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Winding of planar Gaussian processes

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Abstract. We consider a smooth, rotationally invariant, centered Gaussian process in the plane, with arbitrary correlation matrix $C_{tt'}$. We study the winding angle ϕ_t, around its center. We obtain a closed formula for the variance of the winding angle as a function of the matrix $C_{tt'}$. For most stationary processes $C_{tt'} = C(t-t')$ the winding angle exhibits diffusion at large time with diffusion coefficient $D = \int_0^\infty \mathrm{d}s\, C'(s)^2/(C(0)^2 - C(s)^2)$. Correlations of $\exp(in\phi_t)$ with integer n, the distribution of the angular velocity $\dot{\phi}_t$, and the variance of the algebraic area are also obtained. For smooth processes with stationary increments (random walks) the variance of the winding angle grows as $\frac{1}{2}(\ln t)^2$, with proper generalizations to the various classes of fractional Brownian motion. These results are tested numerically. Non-integer n is studied numerically.

Keywords: exact results, stochastic processes (theory), diffusion

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Contents

1. Introduction and model

The winding of planar random processes has been studied for a while. These are of interest for the physics of polymers [1, 2], flux lines in superconductors [3, 4] and the quantum Hall effect [5]. Recently there was revived interest in winding properties of processes described by Schramm–Loewner evolutions (SLE) [6], such as the loop erased random walk [7]. In the case of the planar Brownian motion the distribution of the winding angle ϕ_t around a point O was computed a long time ago by Spitzer [8] who found that $\phi_t/\ln t$ has a Cauchy distribution, i.e. with infinite first moment. This peculiar feature was later understood to be related to the large winding accumulated while the trajectory wanders infinitely close to point O, and is removed by considering either a small excluded region around O, or a lattice cutoff, or some other regularization of the short time, and leading instead to exponentially decaying distributions [9]–[14]. Recently it was found that correlated Gaussian processes related to the fractional Brownian motion, a scale invariant process with stationary increments correlated in time, has similar properties [15].

The aim of this paper is to study the winding of a very general continuous-time Gaussian process $\xi_t = \xi_t^x + \mathrm{i}\xi_t^y$ in the complex plane with arbitrary correlations in time. The only restriction, mainly to avoid a cumbersome formula, is that the measure is rotationally invariant around the origin 0 and the winding angle ϕ_t is measured around point 0, i.e. $\xi_t = r_t \mathrm{e}^{\mathrm{i}\phi_t}$ where ϕ_t is a continuous real function of time and $r_t = |\xi_t|$. The process is thus centered $\langle \xi_t \rangle = 0$ and fully characterized by its two-time correlation function:

$$\langle \xi_t^i \xi_{t'}^j \rangle = \delta_{ij} C_{tt'} \tag{1}$$

with $i, j = x, y$, or equivalently $\langle \xi_t \xi_{t'} \rangle = 0$ and $\langle \xi_t^* \xi_{t'} \rangle = 2C_{tt'}$. The most general form would be $\langle \xi_t^* \xi_{t'} \rangle = 2(C_{tt'} + \mathrm{i}A_{tt'})$ but we also assume reflection symmetry which forbids the antisymmetric term $\epsilon_{ij} A_{tt'}$ in (1) with $\epsilon_{12} = -\epsilon_{21} = 1$. Since the process $\xi_t/\sqrt{C_{tt}}$ has

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the same winding angle as ξ_t, observables involving only the winding angle should only depend on the combination

$$c_{tt'} = C_{tt'}/\sqrt{C_{tt}C_{t't'}} \tag{2}$$

with $c_{tt} = 1$ and from Cauchy–Schwartz inequalities, $|c_{tt'}| \le 1$, the bound being saturated, i.e. $c_{tt'} = \pm 1$, if and only if $\xi_t = \pm\sqrt{(C_{tt}/C_{t't'})}\xi_{t'}$. Some particular cases are (i) the stationary process $C_{tt'} = C(t-t')$ and one defines $c_{t0} = c(t) = C(t)/C(0)$, (ii) the process with stationary increments $C^{(1,1)}_{tt'} := \partial_t\partial_{t'}C_{tt'} = C_2(t-t')$ (here and below we adopt the following definition for partial derivatives: $\partial_t C_{tt'} = C^{(1,0)}_{tt'}$ etc). (Strict) normalizability of the Gaussian measure requires that these functions have (strictly) positive Fourier transforms $\tilde{c}(\omega) \ge 0$ and $\tilde{C}_2(\omega) \ge 0$. In some cases we restrict to a process which everywhere below we call 'smooth', meaning—by definition here— ξ_t differentiable at least once, i.e. $C^{(1,1)}_{tt}$ exists (equal to $-C''(0)$ for a stationary process). For such a smooth process, $\langle \dot{\xi}^i_t \xi^i_t \rangle = C^{(1,0)}_{tt} = C^{(0,1)}_{tt} = \frac{1}{2}\partial_t C_{tt}$, which vanishes if furthermore the process is stationary. These processes appear everywhere in physics; prominent examples occur in the study of quantum noise, where $\tilde{C}(\omega) = \eta\hbar\omega\coth(\hbar\omega/2T)$ (supplemented by a bath-dependent short time cutoff, e.g. at $T=0$, $\tilde{C}(\omega) = \hbar\eta|\omega|\mathrm{e}^{-\tau_0|\omega|}$, i.e. $C(\tau) = (\eta\hbar/\pi)((\tau_0^2-\tau^2)/(\tau_0^2+\tau^2)^2)$) or of quantum Brownian motion [16] and has served as a motivation for the present work [17].

The outline of the paper is as follows. In section 2 we study single-time quantities. The distribution of angular velocity is obtained. In section 3 we study the periodized winding probability distribution which is easier to deal with than the full one. The correlations of $\exp(\mathrm{i}n\phi_t)$ are obtained analytically for integer n, and studied numerically also for non-integer n. In section 4 we obtain a closed formula for the variance of the winding angle as a function of the matrix $C_{tt'}$. We show that for most stationary processes the winding angle exhibits diffusion at large time and we obtain the diffusion coefficient. We also study non-stationary process such as the random walk and the various classes of fractional Brownian motion. Finally in section 5 the variance of the algebraic area is obtained. Most results are tested numerically.

2. Single-time quantities

Single-time quantities are easily extracted from the Gaussian distribution $\sim\mathrm{d}^2\xi_t\,\mathrm{e}^{-|\xi_t|^2/(2C_{tt})}$ by performing a change of variables. Everywhere below $\mathrm{d}^2\xi = \mathrm{d}\xi\,\mathrm{d}\xi^* = \mathrm{d}\xi^x\,\mathrm{d}\xi^y$. The modulus is distributed as $P(r_t)\,\mathrm{d}r_t$ with $P(r) = (r/C_{tt})\mathrm{e}^{-r^2/(2C_{tt})}$; hence the probability of being within $r_t < \epsilon$ near the center vanishes as $\epsilon^2/(2C_{tt})$. To compute the distribution of the angular velocity $\dot{\phi}_t$ one uses that $X_t = (\xi_t, \dot{\xi}_t)$ is Gaussian with measure $(\mathrm{d}^2\xi_t\,\mathrm{d}^2\dot{\xi}_t/(2\pi)^2)\det(M)\mathrm{e}^{-(1/2)X^*MX}$ and correlation matrix $M^{-1} = ((C_{tt}, C^{(1,0)}_{tt}), (C^{(0,1)}_{tt}, C^{(1,1)}_{tt}))$. Let us define $\dot{\xi}_t = \alpha_t\xi_t$ with $\alpha_t = \dot{r}_t/r_t + \mathrm{i}\dot{\phi}_t$. Here we have required a smooth process. The measure becomes $(\mathrm{d}^2\xi_t\,\mathrm{d}^2\alpha_t/(2\pi)^2)|\xi_t|^2\det(M)\mathrm{e}^{-(1/2)\beta|\xi_t|^2}$ where $\beta = (1,\alpha_t^*)M(1,\alpha_t)$. Integration over ξ_t yields the joint distribution $P(\dot{\rho}_t,\dot{\phi}_t)\mathrm{d}\dot{\rho}_t\,\mathrm{d}\dot{\phi}_t$, with $\rho_t = \ln r_t$, equal to

$$\frac{\mathrm{d}\dot{\rho}_t\,\mathrm{d}\dot{\phi}_t}{\pi}\frac{C_{tt}C^{(1,1)}_{tt}}{(C^{(1,1)}_{tt} - 2C^{(1,0)}_{tt}\dot{\rho}_t + C_{tt}(\dot{\rho}_t^2 + \dot{\phi}_t^2))^2}. \tag{3}$$

Integration yields

$$P(\dot{\phi}_t)\,\mathrm{d}\dot{\phi}_t = \mathrm{d}\dot{\phi}_t \frac{a_t}{2(a_t+\dot{\phi}_t^2)^{3/2}} \tag{4}$$

with $a_t = (C_{tt}C_{tt}^{(1,1)} - (C_{tt}^{(1,0)})^2)/C_{tt}^2 = \partial_t\partial_{t'}\ln|c_{tt'}||_{t'=t}$. For a stationary process, $a_t = a = -c''(0)$. For stationary increments, $a_t = C_2(0)/C_{tt} - \frac{1}{4}(\partial_t \ln C_{tt})^2$. Note that this distribution is broad; it does have a first moment but no second moment, i.e. $\langle\dot{\phi}_t^2\rangle$ is infinite.

3. Periodized winding

Next one can compute two-time correlations of the winding angle. The two-time probability measure of the process can be written as

$$\frac{r_t r_{t'}\mathrm{d}r_t\,\mathrm{d}r_{t'}\,\mathrm{d}\phi_t\,\mathrm{d}\phi_{t'}}{(2\pi)^2\Delta_{tt'}}\exp\left(-\frac{C_{t't'}r_t^2 + C_{tt}r_{t'}^2 - 2C_{tt'}r_t r_{t'}\cos(\phi_t-\phi_{t'})}{2\Delta_{tt'}}\right) \tag{5}$$

with $\Delta_{tt'} = C_{tt}C_{t't'} - C_{tt'}^2$; hence integration over r_t and $r_{t'}$ allows us to obtain the probability distribution of $\cos(\phi_t - \phi_{t'})$. Equivalently this gives the probability of $\phi := \phi_t - \phi_{t'}$ modulo 2π, i.e. it gives the periodized probability $\tilde{P}(\phi) = \sum_{m=-\infty}^{+\infty} P(\phi + 2\pi m)$ where $P(\phi)$ is the probability of the total winding $\phi \in]-\infty, +\infty[$. Defining

$$F(z) := \int_{x,y>0} xy\mathrm{e}^{-(x^2/2)-(y^2/2)+xyz} = \frac{1}{1-z^2} + \frac{z\arccos(-z)}{(1-z^2)^{3/2}} \tag{6}$$

for $-1<z<1$, with $\arccos(-z) = \pi - \arccos z$ and $F(z) = \sum_{n=0}^{\infty}(2^n/n!)\Gamma[1+n/2]^2 z^n = 1 + (\pi z/2) + 2z^2 + \mathrm{O}(z^3)$, one finds

$$\tilde{P}(\phi) = \frac{1}{2\pi}(1-c_{tt'}^2)F(c_{tt'}\cos(\phi)). \tag{7}$$

One can check that $\int_0^{2\pi}\mathrm{d}\phi\tilde{P}(\phi) = 1$. An interesting limit is t close to t'. Then $c_{tt'}$ is close to unity and using the expansion $F(z) = (\pi/2\sqrt{2}(1-z)^{3/2}) - (\pi/8\sqrt{2(1-z)}) + \mathrm{O}(1)$ we find that

$$\tilde{P}(\phi) \approx \frac{1-c_{tt'}}{(2(1-c_{tt'})+\phi^2)^{3/2}} \tag{8}$$

and furthermore we expect that $P(\phi)\approx\tilde{P}(\phi)$ since the probability of $2\pi n$ winding with $n \neq 0$ is negligible in that limit. Upon Taylor expansion in $t-t'$ one finds that this result is consistent with (4) but the approach here is more general as the formula (7) does not require a smooth process. The only assumption in (8) is then the continuity of $c_{tt'}$. The short time behavior is controlled by the distribution (8) for moments with $0<\alpha<2$, i.e. $\langle|\phi|^\alpha\rangle_{\tilde{P}} \approx K_\alpha(1-c_{tt'})^{\alpha/2}$ with $K_\alpha = 2^{\alpha/2}\Gamma(1-\alpha/2)\Gamma((1+\alpha)/2)/\sqrt{\pi}$, and becomes dominated by the cutoff at $\phi = \mathrm{O}(1)$ for $\alpha \geq 2$ with $\langle|\phi|^\alpha\rangle_{\tilde{P}} \approx K'_\alpha(1-c_{tt'})^{2-(\alpha/2)}$. For instance the variance of the winding angle is found as

$$\langle\phi^2\rangle_{\tilde{P}} \approx -(1-c_{tt'})\ln(1-c_{tt'}) \tag{9}$$

for $c_{tt'}$ close to unity, and we check below that this coincides with the behavior of $\langle\phi^2\rangle_P$ at short time differences.

This allows us to compute the correlation functions $\mathcal{C}_n(t,t') = \langle e^{in(\phi_t-\phi_{t'})}\rangle$ for *integer* n, which thus have closed expressions as a function of the matrix $C_{tt'}$:

$$\mathcal{C}_n(t,t') = F_n(c_{tt'}). \tag{10}$$

One finds for instance

$$F_1(c) = \frac{1}{c}(E(c^2) + (c^2-1)K(c^2)) \tag{11}$$

$$F_2(c) = 1 + \left(\frac{1}{c^2} - 1\right)\ln(1-c^2) \tag{12}$$

where $E(x) = \int_0^{\pi/2}(1-x\sin^2 y)^{1/2}\,\mathrm{d}y$ and $K(x) = \int_0^{\pi/2}(1-x\sin^2 y)^{-1/2}\,\mathrm{d}y$ are the elliptic integrals. We then obtain the limiting behaviors:

$$\begin{aligned} F_1(c) = \frac{\pi c}{4} + \frac{\pi c^3}{32} + \frac{3\pi c^5}{256} + \mathrm{O}(c^7) = 1 - \frac{1-c}{2}\left(\ln\left(\frac{8}{1-c}\right) - 1\right) \\ + \mathrm{O}(\ln(1-c)(1-c)^2) \end{aligned} \tag{13}$$

for small c (large time separation), and for c near 1 (small time separation), respectively, and $F_2(c) = (c^2/2) + \mathrm{O}(c^4)$ and more generally $F_n(c) = (\Gamma[1+n/2]^2/n!)c^n + \mathrm{O}(c^{n+2})$ for integer n at small c.

We have checked these results numerically for several stationary processes where $c_{tt'} = c(\tau) = C(\tau)/C(0)$ where $\tau = t - t'$. The process ξ_t^i was generated numerically using a discrete Fourier transform of $\sqrt{\tilde{c}(\omega)N\Delta\tau}\mathcal{A}^i$, where N, the number of points, is typically $N = 2^{16}$, $\Delta\tau = 0.01$ is the time segment in the process and $\mathcal{A}^i$ is a unit white Gaussian process. We have computed $\mathcal{C}_n(\tau)$ where the average $\langle e^{in\phi}\rangle$ is over the time range and over several realizations, typically 10. We have plotted $\mathcal{C}_n(\tau)$ parametrically as a function of $c(\tau)$ for various types of noise. Up to numerical accuracy all the curves fall on the predicted master curve $\mathcal{C}_n(\tau) = F_n(c(\tau))$. When $c(\tau)$ is non-monotonic, the master curve may be traced more than once. This is illustrated in figure 1 where we use

$$c(\tau) = \frac{1-\tau^2}{(1+\tau^2)^2} \tag{14}$$

which changes sign and becomes negative before converging to zero: one can distinguish in the inset two branches very close, to within numerical accuracy.

4. Variance of the total winding angle

The previous results are easy to derive, and are simple functions of $c_{tt'}$, but they do not contain information about integer winding. They only probe $\tilde{P}(\phi)$, the periodized winding angle distribution. An interesting question is how to access the full winding distribution $P(\phi)$ and whether its dependence on the matrix $C_{tt'}$ remains tractable. It is a more difficult question, since to compute the full winding angle one must follow somehow the time evolution of the process, e.g. use that $\phi = \phi_t - \phi_{t'} = \int_{t'}^{t}\mathrm{d}\phi_s$. A related difficult question, which requires the full distribution $P(\phi)$, is how to obtain the averages $\mathcal{C}_n(t,t') = \langle e^{in(\phi_t-\phi_{t'})}\rangle$ for *non-integer* n. It is seen in figure 2 that these are not simple functions, but rather unknown and more complicated functionals, of $c_{tt'}$.

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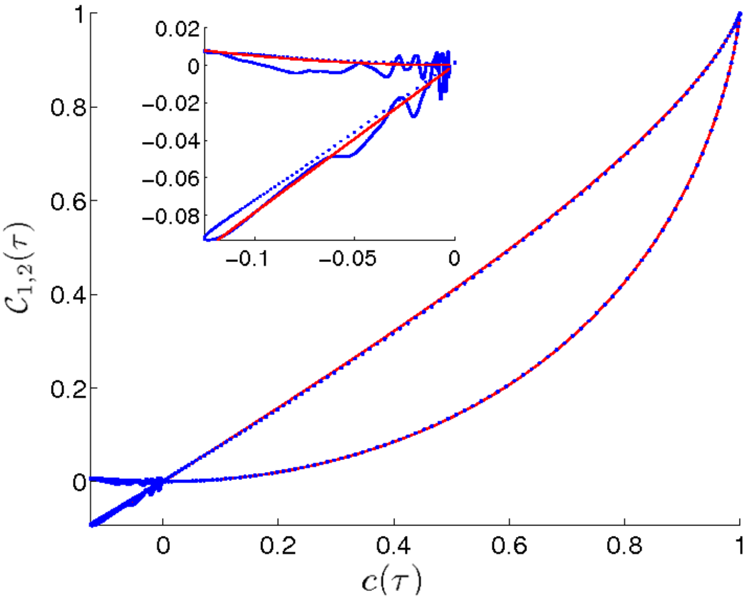

Figure 1. The correlation function $\mathcal{C}_n(\tau)$ for $n = 1$ (top curve) and for $n = 2$ (bottom curve) as a function of $c(\tau)$ for $c(\tau) = ((1-\tau^2)/(1+\tau^2)^2)$ (blue curves) and the prediction for $F_1(c)$ and $F_2(c)$ (red curves). The inset shows how each of these two curves is traced twice for negative values of c (once dotted, once full).

Here we present the simplest result on this question, the variance of the winding angle. Here, for simplicity, and to avoid stochastic calculus subtleties, we first restrict to a smooth, i.e. differentiable process as discussed above. We only need to compute the two-time angular velocity correlation $\mathcal{C}_v(t,t') = \langle \dot\phi_t \dot\phi_{t'} \rangle$. We use that $\dot\phi_t = \mathrm{Im}(\dot\xi \xi^*)/|\xi|^2 = \epsilon_{ij} \xi_t^i \dot\xi_t^j/|\xi_t|^2$; hence,

$$\begin{aligned} \mathcal{C}_v(t,t') &= 2 \int_{s,s'>0} G_{ss'tt'} \\ G_{ss'tt'} &= \langle \dot\xi_t^y \dot\xi_{t'}^y \rangle_{s,s'} \langle \xi_t^x \xi_{t'}^x \rangle_{s,s'} - \langle \dot\xi_t^y \xi_{t'}^y \rangle_{s,s'} \langle \xi_t^x \dot\xi_{t'}^x \rangle_{s,s'} \end{aligned} \tag{15}$$

using isotropy, where $\langle \mathrm{O}[\xi_x] \rangle_{s,s'} = \langle \mathrm{O}[\xi_x] \mathrm{e}^{-s(\xi_t^x)^2 - s'(\xi_{t'}^x)^2} \rangle$ and the same for averages over ξ_y. The integrals over s restore the (difficult) denominators. Each average can be computed from the generating function:

$$\begin{aligned} \langle \mathrm{e}^{\mathrm{i}\mu\xi_t^x + \mathrm{i}\mu'\xi_{t'}^x + \mathrm{i}\mu_1\xi_{t_1}^x + \mathrm{i}\mu_2\xi_{t_2}^x} \rangle &= \mathrm{e}^{-(1/2)(C_{tt}\mu^2 + C_{t't'}\mu'^2) - \mu\mu' C_{tt'}} \\ &\quad \times \mathrm{e}^{-\mu_1\mu_2 C_{t_1 t_2} - \sum_{i=1,2} (1/2) C_{t_i t_i} \mu_i^2 + \mu_i(\mu C_{t_i t} + \mu' C_{t_i t'})} \end{aligned} \tag{16}$$

by taking appropriate derivatives w.r.t. μ_i and t_i at $\mu_i = 0$ and coinciding times, and integrating with the measure $(1/4\pi\sqrt{ss'}) \int_{\mu,\mu'} \mathrm{e}^{-(\mu^2/4s)-((\mu')^2/4s')}$ to restore the $\langle \cdots \rangle_{s,s'}$ averages. After straightforward but lengthy algebra, and massive simplifications, one finds

$$G_{ss'tt'} = \frac{-C_{tt'}^{(1,0)} C_{tt'}^{(0,1)} + C_{tt'}^{(1,1)} C_{t,t'}}{(1 + 2sC_{tt} + 2s'C_{t't'} + 4ss'(C_{tt}C_{t't'} - C_{tt'}^2))^2}$$

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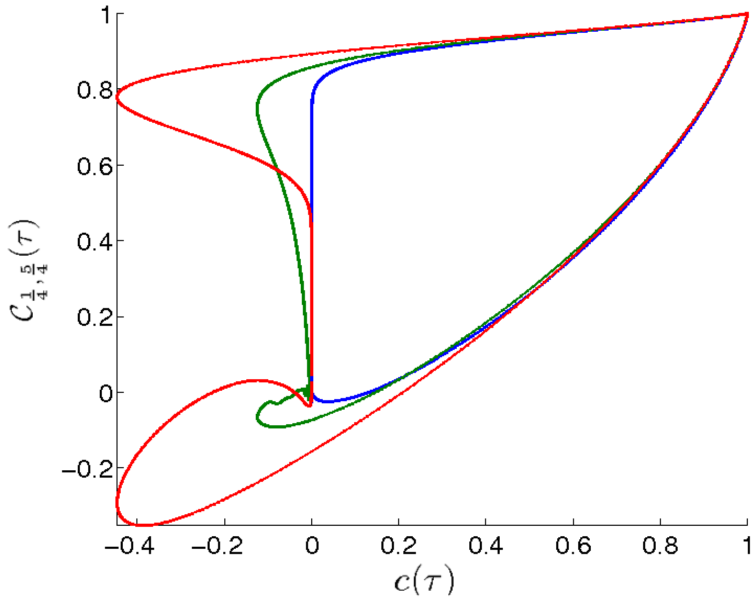

Figure 2. The correlation function $\mathcal{C}_n(\tau)$ for $n = \frac{1}{4}$ (three curves starting at the top), $\frac{5}{4}$ (three curves starting at the bottom) as a function of $c(\tau)$ for three processes: $c(\tau) = \exp -\tau^2/2$ (in blue), the $c(\tau)$ given by (14) used in figure 1 (in green), $c(\tau) = (1-\tau^2)\exp -\tau^2/2$ (in red). Note that for each n the three curves remain very close for $c > 0.4$ and that for $n = 5/4$ all processes change sign.

which, after integration over s, s', gives the angular velocity correlation, which is our main result:

$$\mathcal{C}_v(t,t') := \langle \dot\phi_t \dot\phi_{t'} \rangle = \frac{1}{2}\left(\frac{-C^{(1,1)}_{tt'} C_{tt'} + C^{(1,0)}_{tt'} C^{(0,1)}_{tt'}}{C^2_{tt'}} \right) \ln(1 - c^2_{tt'})$$
$$= -\tfrac{1}{2}(\partial_t \partial_{t'} \ln |c_{tt'}|) \ln(1 - c^2_{tt'}) \tag{17}$$

where we recall that $c_{tt'} = C_{tt'}/\sqrt{C_{tt} C_{t't'}}$. The variance of the winding angle is then obtained as

$$\Phi_{tt'} = \langle (\phi_t - \phi_{t'})^2 \rangle = \int_{t'}^{t} \mathrm{d}t_1 \int_{t'}^{t} \mathrm{d}t_2\, \mathcal{C}_v(t_1, t_2) \tag{18}$$

hence $\partial_t \Phi_{tt'} = 2 \int_{t'}^{t} \mathrm{d}t_2\, \mathcal{C}_v(t, t_2)$, where $\mathcal{C}_v(t_1, t_2)$ is given by (17). We now discuss separately stationary and non-stationary processes.

4.1. Stationary processes

Let us study first stationary processes $c_{tt'} = c(\tau) = C(\tau)/C(0)$ with $\tau = t - t'$. Then the angular velocity correlation becomes $\mathcal{C}_v(t,t') = \mathcal{C}_v(t - t')$ with

$$\mathcal{C}_v(\tau) = \frac{1}{2}\left(\frac{C''(\tau)C(\tau) - C'(\tau)^2}{C(\tau)^2} \right) \ln(1 - c(\tau)^2) = \frac{1}{2}(\partial^2_\tau \ln |c(\tau)|) \ln(1 - c(\tau)^2) \tag{19}$$

which exhibits a divergence at small time τ, $\mathcal{C}_v(\tau) \approx c''(0)\ln(\tau\sqrt{-c''(0)})$, but an integrable one. The winding angle variance takes the form $\Phi_{tt'} = \Phi(t-t')$, and using that $\partial_\tau \Phi(\tau) = 2\int_0^\tau \mathrm{d}\tau_2\, \mathcal{C}_v(\tau_2)$ one finds upon integration by parts

$$\partial_\tau \Phi(\tau) = 2\int_0^\tau \mathrm{d}s \frac{c'(s)^2}{1-c(s)^2} + \frac{c'(\tau)}{c(\tau)}\ln(1-c(\tau)^2) \tag{20}$$

with no boundary term at $\tau = 0$ since $c(\tau) \approx 1 + \frac{1}{2}c''(0)\tau^2$ at small τ, with $c''(0) < 0$ since the process is smooth. The small τ behavior of the variance of the winding angle is $\Phi(\tau) \approx \tau^2(c''(0)\ln(\tau\sqrt{-c''(0)}) - \frac{3}{2}c''(0))$. More generally, the formula (20) holds for processes such as $c(\tau) = \mathrm{e}^{-\tau^a}$ at small τ with $a > 1$, so the small time singularity of $\mathcal{C}_v(\tau)$ is integrable.

If we now consider processes such that $c(+\infty) = 0$ then we find that the generic behavior is that the winding angle *diffuses* at large time as $\Phi(\tau) \sim 2D\tau$ with a diffusion coefficient:

$$D = \int_0^\infty \mathrm{d}s \frac{c'(s)^2}{1-c(s)^2} \tag{21}$$

an integral which converges at small s when the process is smooth since then $c'(0) = 0$. In fact, this formula, as well as (20), holds also for some processes with $c''(0) = +\infty$, e.g. such as $c(\tau) = \mathrm{e}^{-\tau^a}$ at small τ with $a > 1$, the main condition being that the small time singularity of $\mathcal{C}_v(\tau)$ is integrable. The convergence at large s should be discussed separately. Since $D > \int \mathrm{d}s\, c'(s)^2$ a necessary condition for convergence at large s is $\int \mathrm{d}s\, c'(s)^2 = \int (\mathrm{d}\omega/2\pi)\omega^2 \tilde{c}(\omega)^2 < +\infty$. For a positive $c(s)$ this is guaranteed by $c(\infty) = 0$. For oscillating $c(s)$, e.g. $c(s) = \cos(s)f(s)$, it requires $|f(s)|$ to decay faster than $1/\sqrt{s}$, or in the Fourier case, e.g. if $c(\omega) \sim 1/|\omega-\omega_c|^b$, then one must have $b < 1$. Interestingly, (21) can also be written as

$$D = \int_0^\infty \mathrm{d}s \left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 \tag{22}$$

where $c(s) = \sin\theta(s)$ with $\theta(s) \in [-\pi/2, \pi/2]$, $\theta(0) = \pi/2$ and here $\theta(+\infty) = 0$. From there one sees that the strict criterion for convergence is that $\int (\mathrm{d}\omega/2\pi)\omega^2 c_\theta(\omega)^2 < +\infty$ where $\tilde{c}_\theta(\omega)$ is the Fourier transform of $c_\theta(\tau) = \langle\theta(0)\theta(\tau)\rangle$, and is also positive (the variable θ however is not Gaussian).

Let us give examples of some of the non-generic situations where winding angle diffusion does not occur. The simplest case is the process $\xi_t = \xi_1 \mathrm{e}^{\mathrm{i}t} + \xi_2 \mathrm{e}^{-\mathrm{i}t}$ with ξ_1 and ξ_2 two i.i.d. complex Gaussian noises, which has correlation $c_{tt'} = \cos(t-t')$. One finds $C_v(\tau) = -(\ln\sin^2\tau)/(2\cos^2\tau)$ and $\partial_\tau\Phi(\tau) = 2\tau - \tan\tau\ln(\sin^2\tau)$; hence the winding angle grows faster than diffusively as $\Phi(\tau) \sim \tau^2$. Consider next $c(\tau) = J_0(\tau)$, where $J_0(z)$ is the Bessel function, i.e. in the Fourier case $c(\omega) \sim \theta(1-\omega^2)(1-\omega^2)^{1/2}$. The integral (21) is log-divergent at large s and one finds superdiffusion $\Phi(\tau) \sim (2/\pi)\tau\ln\tau$ at large τ. A range of superdiffusion can be obtained, e.g. $c(\tau) \sim \tau^{-b}\sin(\tau+\psi)$ at large τ yields $\Phi(\tau) \sim \tau^{2-2b}$ for $0 < b < 1/2$.

The above predictions are checked numerically in figure 3 in the time variable τ, and as a parametric plot using $c(\tau)$ in figure 4, for the diffusive and superdiffusive cases.

Finally let us consider the stationary process $c_a(\tau) = \mathrm{e}^{a-\sqrt{a^2+\tau^2}}$, which as $a \to 0$ converges to the non-smooth process $C(\tau) = \mathrm{e}^{-|\tau|}$. For any small $a > 0$ the process

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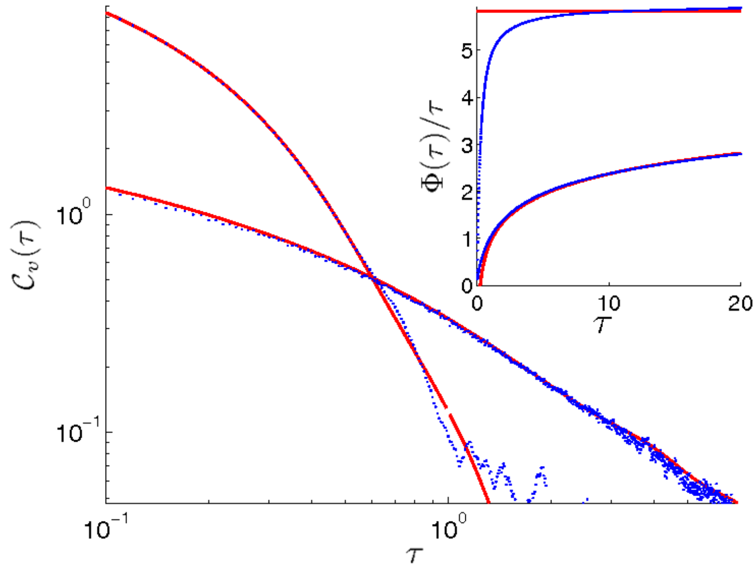

Figure 3. The angular velocity correlation function $\mathcal{C}_v(\tau)$ as a function of τ for the $c(\tau)$ used in figure 1 (the blue curve with the stronger decay), and for $c(\tau) = J_0(\tau)$ (the second blue curve), together with the predictions of equation (19) (red curves). In the inset the winding angle variance $\Phi(\tau)$, divided by τ, is displayed for the same two cases. From the top, the first function (in blue) saturates to its diffusive value (red line), with $D \sim 2.92$ calculated from equation (21). The second function (in blue) is compared with the superdiffusion prediction $\Phi(\tau) = (2/\pi)\tau \log\tau + 0.907\tau$ from equation (20). Both results are averages over 50 realizations.

is smooth and leads to diffusion. The diffusion coefficient however diverges as $D \approx \frac{1}{2}\ln(1/a)+(\pi/4)$ as $a \to 0$. As we will see below, the non-smooth limit process $C(\tau) = \mathrm{e}^{-|\tau|}$ is related to Brownian motion and leads to broad distributions of winding angle and to an infinite variance due to singular behavior at short times.

4.2. Non-stationary processes

We now study non-stationary processes. Such processes often occur in the context of ageing or coarsening dynamics [18]. In fact they can, in some cases, be mapped onto a stationary process using the property of reparametrization of time: if the process $c_{tt'}$ has a winding angle ϕ_t then the process $c_{g(t)g(t')}$ has a winding angle $\phi_{g(t)}$ for any positive monotonic function $g(t)$. Note that equations (17) and (18) have precisely the differential form required to satisfy this property. Hence for processes of the form $c_{tt'} = \hat{c}(g(t)-g(t'))$, the variance of the winding angle is immediately obtained as $\Phi_{tt'} = \hat{\Phi}(g(t) - g(t'))$ where $\hat{\Phi}(\tau)$ is the variance for the stationary process $\hat{c}(\tau)$. Hence diffusion in $\hat{\Phi}(\tau) \sim 2D\tau$ implies $\Phi_{tt'} = 2(g(t)-g(t'))D$ at well separated times. One example, frequent in ageing processes, is $c_{tt'} = f(t'/t)$ for $t > t'$. Then one can choose $g(t) = \ln t$ and $\hat{c}(s) = f(\mathrm{e}^{-s})$. To avoid divergences at short time difference one must have that $f(x) = 1 - (1 - x)^b$ for x close to unity, with $b > 1$. One example is $f(x) = \mathrm{e}^{-|\ln x|^b}$. If this is the case, and provided that we have convergence at large s, one finds that $\Phi_{tt'} \sim 2D\ln(t/t')$ at well separated times,

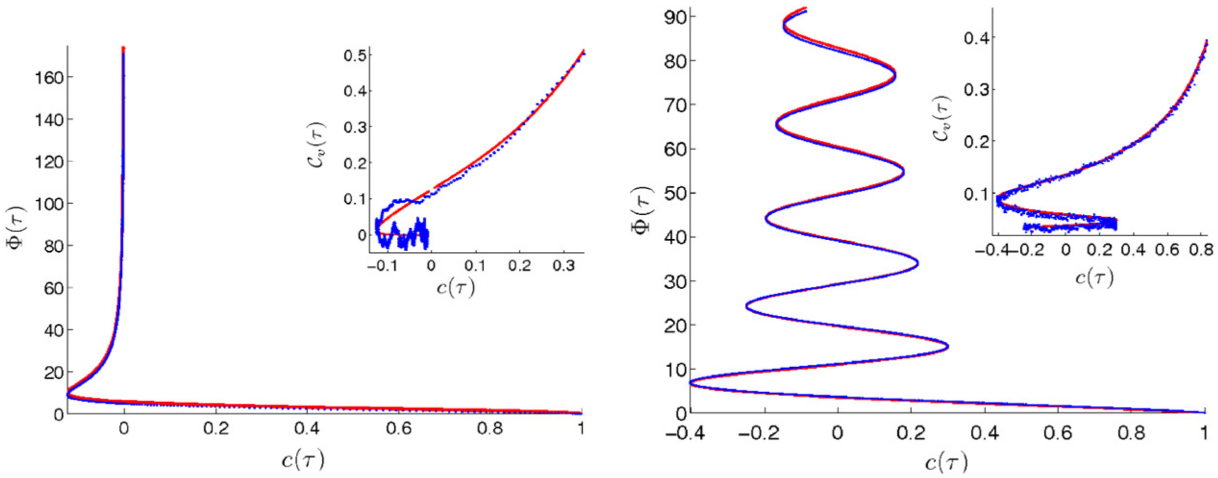

Figure 4. Left panel: parametric plot of the winding angle variance $\Phi(\tau)$ (y-axis) and $c(\tau)$ (x-axis) for the $c(\tau)$ used in figure 1 (blue curves) and the linear diffusion formula $\Phi(\tau) = 2D\tau$ with $D \sim 2.92$ as predicted from equation (21) (in red). The inset shows the angular velocity correlation function $\mathcal{C}_v(\tau)$ as a function of $c(\tau)$ for the same choice of $c(\tau)$ (blue curves) and the results of equation (19) (red curves). Right panel: parametric plot of the winding angle variance $\Phi(\tau)$ as a function of $c(\tau)$ for $c(\tau) = J_0(\tau)$ (blue curves) and the asymptotic prediction $\Phi(\tau) = (2/\pi)\tau \log \tau + 0.907\tau$ calculated from equation (20) (in red). In the inset the correlation function $\mathcal{C}_v(\tau)$ is shown as a function of $c(\tau)$ for the same $c(\tau)$ (blue curves), together with the results of equation (19) (red curves). Both results are averages over 50 realizations.

i.e. diffusion in the logarithm of time. Clearly the case $b = 1$ leads again to a non-smooth process and is discussed below.

Among non-stationary processes, processes with stationary increments are of special importance. One such process is the so-called fractional Brownian motion (FBM), $C_{tt'} = \frac{1}{2}(t^{2h} + (t')^{2h} - |t - t'|^{2h})$, with $0 < h < 1$, which is the only member of this class which is also scale invariant. For $h = 1/2$ one recovers the standard Brownian motion. The FBM with $h > 1/2$ is sufficiently smooth for the above considerations to apply and one easily sees that the time change $g(t) = \ln t$ and

$$\hat{c}(s) = \cosh(hs) - 2^{2h-1}|\sinh(s/2)|^{2h} \tag{23}$$

can be used, leading to diffusion for the winding angle in the variable $g(t) = \ln t$ at large times, i.e. $\Phi_{tt'} \sim 2D_h \ln(t/t')$ where $D_h = \int_0^\infty \hat{c}'(s)^2/(1 - \hat{c}(s)^2)$ diverges as $h \to 1/2^+$.

The cases of the Brownian motion $h = 1/2$ and of the FBM for $h < 1/2$ require a separate discussion. Let us first recall what was found for the two-dimensional BM with diffusion coefficient D_0, i.e. $\langle|\xi_t|^2\rangle = 2C_{tt} = 2D_0 t$. At large t, for BM in the full plane the classical result [8] is that $y = 2\phi_t/\ln t$ has a Cauchy distribution $p_0(y) = (1/\pi)(1 + y^2)^{-1}$; hence the variance $\Phi_{tt'}$ cannot be defined. This broad distribution is regularized in the presence of an absorbing small disk centered at 0 of radius R, where the distribution of $y = 2\phi_t/\ln(D_0 t/R^2))$ becomes [10] $p_A(y) = \pi/(4\cosh^2(\pi y/2))$. For a reflecting small disk (or for big windings around several point centers or for a lattice random walk around a point not on the lattice) one finds [11] another distribution, $p_R(y) = 1/(2\cosh(\pi y/2))$. In the latter two cases one has $\Phi_{tt'} \sim \frac{1}{12}\ln^2(D_0 t/R^2)$ at large t and fixed t' for absorbing obstacles (resp. $\Phi_{tt'} \sim \frac{1}{4}\ln^2(D_0 t/R^2)$ for reflecting ones); however these are no longer Gaussian processes, so the comparison with our results is not straightforward. We see below however that the process studied here yields some similar result. Before we do so let us recall how the above scaling $\phi_t \sim \ln t$ for the (regularized) BM results can be understood from simple arguments. From the BM properties one easily obtains the stochastic equations for radius and angle (in the Ito formulation) as $\mathrm{d}r_t = \mathrm{d}B_t + (\mathrm{d}t/2r_t)$ and $\mathrm{d}\phi_t = \mathrm{d}\tilde{B}_t/r_t$ where, for $D_0 = 1$, B_t, $\tilde{B}_t$ are two independent unit Brownian motions (i.e. with $\langle \mathrm{d}B_t^2\rangle = \langle \mathrm{d}\tilde{B}_t^2\rangle = \mathrm{d}t$). Diffusion in the winding angle is thus only possible if $\langle 1/r_t^2\rangle$ is bounded, as also nicely discussed in [14]. If the Brownian can explore large distance then $\langle 1/r_t^2\rangle \sim \ln(D_0 t/R^2)/(D_0 t)$, where the small distance cutoff is also necessary, and one recovers the above non-diffusive behavior $\phi_t \sim \ln t$ from the estimate $\mathrm{d}\phi_t^2 = D_0\,\mathrm{d}t\langle 1/r_t^2\rangle \sim \mathrm{d}t\ln(D_0 t/R^2)/t$.

Can we make contact with our results, in particular can we also obtain from our formula (17) the (regularized) BM scaling $\Phi_{tt'} \sim \ln^2 t$? The answer is yes, but since we can only address smooth processes, we now consider the general smooth process with stationary increments:

$$C_{tt'} = \tfrac{1}{2}(f(t) + f(t') - f(t-t')) \tag{24}$$

with $f''(t) = 2C_2(t)$ in the notation of section 1, with $f(0) = 0$ hence $C_{tt} = f(t)$. The choice $f(t) \sim t$ at large t corresponds to the random walk with a short time cutoff. Apart from the short times, it should look like the BM on large timescales. One example is $\xi_t = \int_0^t \mathrm{d}t' \eta_{t'}$ where $\langle \eta_t \eta_{t'}\rangle = \frac{1}{2}\mathrm{e}^{-|t-t'|}$; then $f(t) = t - 1 + \mathrm{e}^{-t} \sim t^2/2$ at short times. In general $f(t)$ is an increasing function. Taking the large t limit at fixed $\tau = t - t_2$, one finds

$$\mathcal{C}_v(t, t-\tau) \approx -\frac{f''(\tau)}{2f(t)}\ln(f(\tau)/f(t)). \tag{25}$$

This gives $\partial_t \Phi_{tt'} = 2\int_0^{t-t'} \mathrm{d}\tau\, \mathcal{C}_v(t, t-\tau) \approx (\ln f(t)/f(t)) \int_0^\infty \mathrm{d}\tau f''(\tau)$; hence for the random walk $f(t) \sim D_0 t$ at large t one finds

$$\Phi_{tt'} \sim \tfrac{1}{2}(\ln t)^2 \tag{26}$$

and one recovers the behavior of the regularized BM. The prefactor, however, is different from both the absorbing and reflecting core ones, but is nicely anticipated from the simple argument presented above. This prefactor is checked numerically in figure 5 for the random walk. In this figure the result for the FBM with $h = 0.6$ is also shown, and exhibits faster convergence to the asymptotics than the random walk since for $h > 1/2$ no small time regularization is required.

The same calculation can be performed for the regularized FBM for $h < 1/2$, i.e. for the FBM random walk with $f(t) \sim t^{2h}$ at large t with $h < 1/2$ and $f(t)$ smooth at small t. There one finds that the leading term above vanishes and one obtains

$$\Phi_{tt'} \sim \frac{t^{1-2h}}{2(1-2h)}\int_0^\infty \mathrm{d}\tau \frac{f'(\tau)^2}{f(\tau)}, \tag{27}$$

i.e. a much faster growth of the variance of the winding angle.

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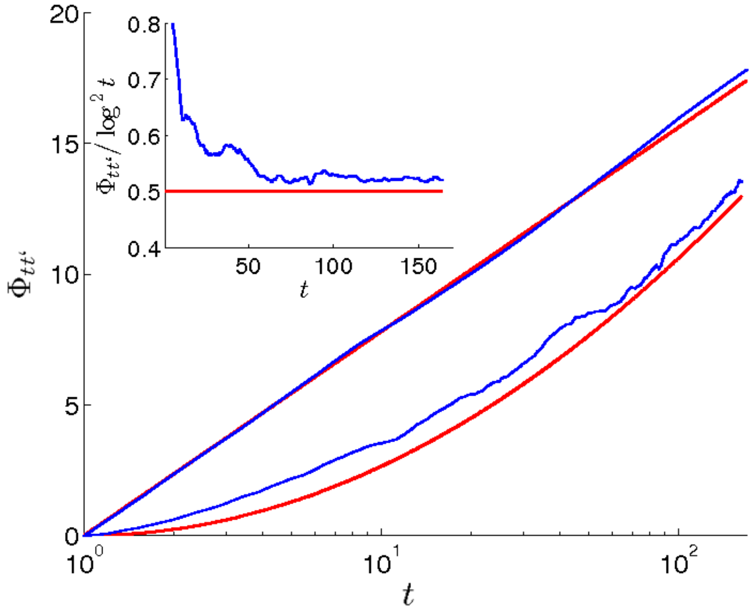

Figure 5. The variance of the winding angle $\Phi_{tt'}$ for $t' = 1$ as a function of t for the random walk, i.e. $C_{tt'}$ of equation (24) with $f(t) = t-1+\mathrm{e}^{-t}$ (bottom blue curve) as compared to the asymptotic prediction of equation (26) (the corresponding red curve). The correlation functions $\Phi_{tt'}$ for $t' = 1$ as a function of t for the FBM with $h = 0.6$ calculated using the equivalent stationary process (23) with the time reparametrization $s \to \mathrm{e}^t$ (top blue curve). The asymptotic diffusion prediction, $\Phi_{tt'} = 2D_{h=0.6} \log t$, where equation (21) gives $D_{h=0.6} \approx 1.7$ is also shown (top red curve). The results for the random walk required an averaging over $\sim 10^3$ realizations. Inset: the ratio $\Phi_{tt'}/\ln^2 t$ for the random walk, for the same data, showing the convergence towards 1/2 as predicted in (26).

Note that apart from in the case of exact scale invariance $f(t) = t^{2h}$, there is no time reparametrization which allows us to map the problem (24) with an arbitrary $f(t)$ to a stationary process. And only for $h > 1/2$ is the stationary process corresponding to the FBM, $f(t) = t^{2h}$, smooth enough that the present results can be used: note that in that case, the contribution (25) of the regime of fixed $t - t_2$ is integrable and contributes only a constant to the winding variance, while the regime of t_2/t fixed (usually called the ageing regime when dealing with two-time correlations) gives the main contribution, leading to the result $\Phi_{tt'} \sim 2D_h \ln(t/t')$ found above. Conversely, the ageing regime gives exactly zero contribution for the BM $h = 1/2$. Indeed then $C_{tt'} = t+t'-|t-t'|$; hence $c_{tt'} = \sqrt{t'/t}$ for $t > t'$, and $C_v(t, t') = 0$ for $t > t'$ with a singularity at $t = t'$. Similarly, the ageing regime is subdominant for the FBM random walk with $h \leq 1/2$.

4.3. From the Spitzer result to the winding of the stationary process $c(\tau) = \mathrm{e}^{-|\tau|}$

Let us return to the unrestricted BM motion for which $C_{tt'} = \min(t, t')$; hence $c_{tt'} = \sqrt{t'/t}$ for $t > t'$. Clearly because of the singularity at $t = t'$ in $C_v(t, t')$ one cannot apply our formula for the winding angle to this case. However the formula (10) does hold for the

BM and one has for integer n

$$\langle \mathrm{e}^{\mathrm{i}n(\phi_t-\phi_{t'})}\rangle = F_n(\sqrt{t/t'}) \tag{28}$$

with the large t decay (at fixed t') $\langle \mathrm{e}^{\mathrm{i}n(\phi_t-\phi_{t'})}\rangle \sim (\Gamma[1+n/2]^2/n!)(t'/t)^{n/2}$.

Using the time change $g(t) = \ln t$ one can now use the reverse correspondence and transfer the Spitzer result [8] for planar BM to the stationary process $\hat{c}(\tau) = \mathrm{e}^{-(1/2)|\tau|}$ which is not smooth and cannot be analyzed with the above methods. Hence the prediction for this process is that $y = 2\phi_\tau/\tau$ is distributed at large τ with the Cauchy distribution $p_0(y)$. Its variance $\Phi_{tt'}$ is thus infinite at all times, as for the BM.

An example of such a process is the Brownian motion or an ideal chain in a harmonic well, with $\tilde{C}(\omega) = 1/(\omega^2+m^2)$; hence in real time $C(\tau) = (1/2m)\mathrm{e}^{-m|\tau|}$ and $c(\tau) = \mathrm{e}^{-m|\tau|}$. The distribution of the winding for such a confined Brownian is thus again the unit Lorentzian distribution for the scaled variable $y = \phi_\tau/m\tau$. An interesting generalization of a discrete version of this model to a chain was studied in [19], with the result that, again, each monomer sees a Lorentzian winding.

It is interesting to now consider a smoother variant of this model, i.e. an ideal chain with a small curvature energy in a harmonic well, described by $\tilde{C}(\omega) = (\omega^2+m^2)^{-1} - (\omega^2+M^2)^{-1} \approx 1/(\omega^4/M^2+\omega^2+m^2)$ at large $M \gg m$. The decay in the time domain, $c(\tau) = (M\mathrm{e}^{-m\tau} - m\mathrm{e}^{-M\tau})/(M-m)$ is now smooth at small times, and one finds that the winding angle recovers now a finite variance and is diffusive, $\Phi_{tt'} \sim 2D\tau$, with a diffusion coefficient $D \sim \frac{1}{2}M\ln(4.2/M)$.

5. Algebraic area enclosed

Finally we can study the algebraic area A_t enclosed by the process, which satisfies $\dot{A}_t = \frac{1}{2}(\xi^x_t\dot{\xi}^y_t - \xi^y_t\dot{\xi}^x_t)$. Its variance is extracted from $G_{0,0,t,t'}$ above and one easily finds that

$$\mathcal{C}_A(t,t') = \langle \dot{A}_t\dot{A}_{t'}\rangle = \tfrac{1}{2}(C^{(1,1)}_{tt'}C_{t,t'} - C^{(1,0)}_{tt'}C^{(0,1)}_{tt'}). \tag{29}$$

For a smooth stationary process one finds $\mathcal{C}_A(\tau) = -\frac{1}{2}C''(\tau)C(\tau) + \frac{1}{2}C'(\tau)^2$, and $\partial_\tau\langle A^2(\tau)\rangle = 2\int_0^\tau \mathcal{C}_A(\tau) = 2\int_0^\tau C'(s)^2\,\mathrm{d}s - C'(\tau)C(\tau)$ and one finds the diffusion result $\langle A^2(\tau)\rangle \sim 2D_A\tau$ with $D_A = \int_0^\infty C'(s)^2\,\mathrm{d}s$. Let us consider now the above process with stationary increments. Note that the time reparametrization is useless here. For the random walk $f(t) \sim D_0 t$ one finds $\langle A_t^2\rangle \sim D_0 t^2/4$ at large t. This is larger than the result for Brownian paths constrained to come back to their starting points (loops) obtained in [14]. This is well confirmed by our numerics displayed in figure 6, where the result for a stationary process, which instead exhibits only diffusive growth of the area, is also shown for comparison.

6. Conclusion

We have computed here the angular velocity correlation of a very general smooth Gaussian process in the plane. This allowed us to obtain a simple closed formula for the diffusion coefficient of the winding angle valid for most such stationary processes. Our formula also extends to non-stationary processes, and has allowed us to obtain the three main

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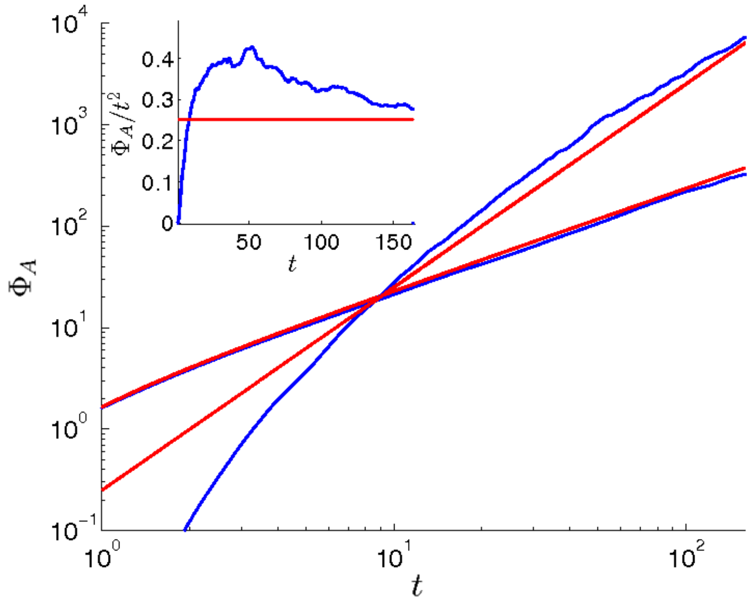

Figure 6. The variance of the algebraic area $\Phi_A = \langle [A_t - A_{t'}]^2 \rangle$ with $t' = 1$ as defined above, equation (29), for: (i) the random walk, i.e. $C_{tt'}$ as in figure 5 (blue curve at the top at large time) and the asymptotic prediction $\sim t^2/4$ (the corresponding red curve); (ii) the result for the stationary process $c(\tau)$ of figure 1 as a function of $t \equiv \tau$ (bottom curve) and the asymptotic prediction $2D_A t$ with $D_A = 3\pi/8$ (the corresponding red curve). In the inset the ratio Φ_A/t^2 is plotted for the random walk as a function of t, and shows saturation towards the predicted prefactor 1/4.

behaviors: (i) diffusion in the logarithm of time for sufficiently smooth fractional Brownian motion; (ii) the square of the logarithm in time (26) for the winding for the usual random walks related to the Brownian motion; (iii) power law growth in time (27) for the winding angle of the random walks which provide a regularization of the non-smooth fractional Brownian motion.

Various extensions of the present calculations are left for the future. These include: higher moments and distributions of area and winding, winding around a point different from the origin, or around several points, winding conditioned to closed paths, the most general 2D Gaussian process including a non-zero average, and finally, devising methods for accounting perturbatively for non-Gaussian effects. It would also be interesting to study persistence effects such as the probability that the winding angle never crosses zero, or to compute the winding for loops, i.e. conditioning the process to return to its starting point.

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References

[1] de Gennes P G, 1971 *J. Chem. Phys.* **55** 572
Rudnick J and Hu Y, 1988 *Phys. Rev. Lett.* **60** 712
Rudnick J and Hu Y, 1987 *J. Phys. A: Math. Gen.* **20** 4421
[2] Grosberg A and Frisch H, 2003 *J. Phys. A: Math. Gen.* **36** 8955 [arXiv:cond-mat/0306586]
[3] Nelson D R, 1988 *Phys. Rev. Lett.* **60** 1973
[4] Drossel B and Kardar M, 1996 *Phys. Rev.* E **53** 5861
[5] Shapere A and Wilczek F, 1990 *Geometric Phases in Physics* (Singapore: World Scientific)
Comtet A, Georgelin Y and Ouvry S, 1989 *J. Phys. A: Math. Gen.* **22** 3917
[6] Schramm O, 2000 *Isr. J. Math.* **118** 221
Cardy J, 2005 *Ann. Phys.* **318** 81
[7] Hagendorf C and Le Doussal P, 2008 *J. Stat. Phys.* **133** 231 [arXiv:0803.3249]
[8] Spitzer F, 1958 *Trans. Am. Math. Soc.* **87** 187
[9] Messulam P and Yor M, 1982 *J. Lond. Math. Soc.* **26** 348
Fisher M, Privman V and Redner S, 1984 *J. Phys. A: Math. Gen.* **17** L569
[10] Rudnick J and Hu Y, 1987 *J. Phys. A: Math. Gen.* **20** 4421
Saleur H, 1994 *Phys. Rev.* E **50** 1123
[11] Pitman J W and Yor M, 1986 *Ann. Probab.* **14** 733
Belisle C, 1989 *Ann. Probab.* **17** 1377
[12] Duplantier B and Saleur H, 1988 *Phys. Rev. Lett.* **60** 2343
[13] Pitman J and Yor M, 1989 *Ann. Probab.* **17** 965
Samokhin K, 1998 *J. Phys. A: Math. Gen.* **31** 8789
Samokhin K, 1998 *J. Phys. A: Math. Gen.* **47** 9455
[14] Comtet A, Desbois J and Ouvry S, 1990 *J. Phys. A: Math. Gen.* **23** 3563
Comtet A, Desbois J and Monthus C, 1993 *J. Stat. Phys.* **73** 433
[15] Rajabpour M A, 2008 *J. Phys. A: Math. Theor.* **41** 425001
[16] Fisher M P A and Zwerger W, 1985 *Phys. Rev.* B **32** 6190
Zwerger W, 1987 *Phys. Rev.* B **35** 4737
Feynman R P and Vernon F L, 1963 *Ann. Phys., NY* **24** 118
[17] Horovitz B, Etzioni Y and Le Doussal P, 2009 in preparation
[18] see e.g. Cugliandolo L F, Kurchan J and Parisi G, 1994 *J. Physique* **4** 1641 [arXiv:cond-mat/9406053]
Bray A J, 1994 *Adv. Phys.* **43** 357 [arXiv:cond-mat/9501089]
[19] Benichou O and Desbois J, 2000 *J. Stat. Phys.* **101** 921 [arXiv:cond-mat/0005156]
Benichou O and Desbois J, 2000 *J. Phys. A: Math. Gen.* **33** 6655 [arXiv:cond-mat/0005155]

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