The Lognormal-Like Statistics of a Stochastic Squeeze Process

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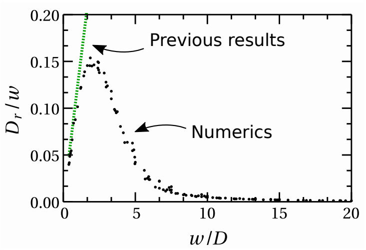
Ben Gurion University

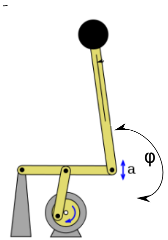


[1] D. Shapira and D. Cohen (Phys. Rev. E, 2017)

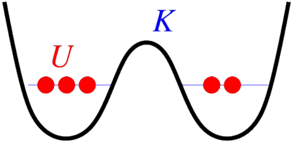
Motivation

Stabilize an inverted pendulum ($\varphi = \pi$):

- Periodic driving (Kapitza)
- Noisy driving



Stabilize condensate in BHH



G. Gordon and G. Kurizki Phys. Rev. Lett. (**2006**).
Y. Khodorkovsky, G. Kurizki, A. Vardi Phys. Rev. Lett. (**2008**), Phys. Rev. (**2009**).
C. Khripkov, A. Vardi, D. Cohen Phys.Rev. A (**2012**).

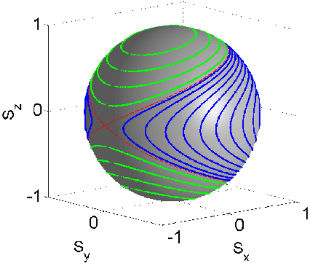
Bose-Hubbard dimer is like a pendulum

$$\mathcal{H} \;=\; \frac{U}{2}\sum_{i=1,2}\hat{n}_i(\hat{n}_i-1) - \frac{K}{2}(\hat{a}_2^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_2)$$

N particles in a double well is like spin $j = N/2$ system

$$\mathcal{H} \;=\; U\hat{J}_z^2 \;-\; K\hat{J}_x$$

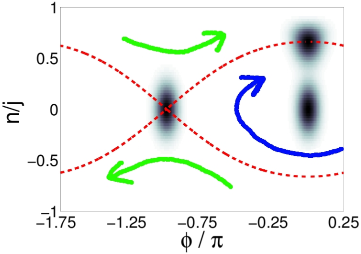
$\hat{J}_z =$ occupation difference



Analogous to Josephson junction if the occupation difference $\ll N/2$

$$\mathcal{H}(n,\varphi) \;=\; Un^2 - \frac{NK}{2}\cos(\varphi)$$

$\hat{n} =$ occupation difference



- Condense all bosons upper orbital (π state) $\sim (a_1^\dagger - a_2^\dagger)^N|0\rangle$

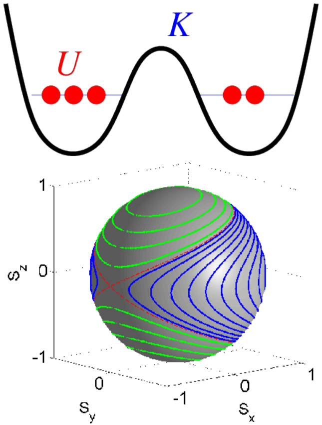
Stabilizing using noisy driving

- Condense all bosons upper orbital (π state)
- Stabilizing: noisy driving (QZE):

$$\mathcal{H} = U\hat{J}_z^2 - [K + \Omega(t)]\hat{J}_x$$

Dynamics near the π-point

- Gaussian around hyperbolic point
- Equations of motion



The model

- Stochastic Differential Equation in 2D
- 2-parameter* model: w, D
- Langevin equation (Stratonovich):

$$
\begin{aligned}
\dot{x} &= \; wx - \Omega(t)y \\
\dot{y} &= -wy + \Omega(t)x \\
\langle \Omega(t)\Omega(t') \rangle &= 2D\delta(t-t')
\end{aligned}
$$

- x, y coordinates $\sim$ major axes of the hyprebolic point

* scaling time: $t \rightarrow wt$. One parameter (w/D).

The model

Langevin equation (Stratonovich):

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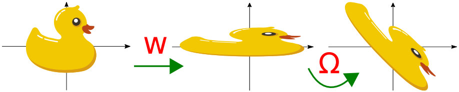

w generates a squeeze: $x = x_0 e^{wt}, \; y = y_0 e^{-wt}$
$\log(r) \sim wt$

$\Omega(t)$ generates rotations
$\log(r) \sim$ const

Interplay between w and D
$\log(r) \sim$ Diffusive spreading (D_r) with drift (w_r)

$r^2 = x^2 + y^2$

The model

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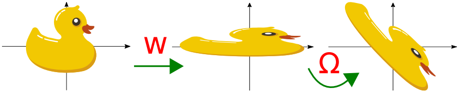

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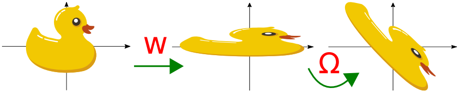

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Interplay between w and D
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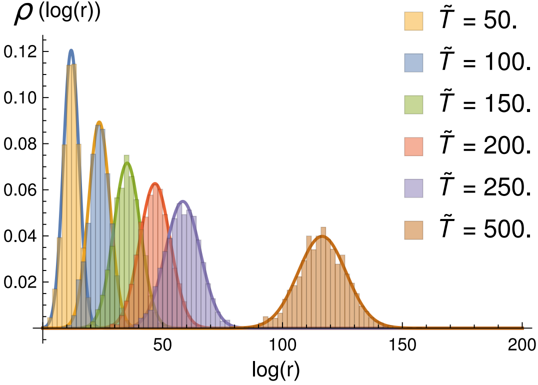
$r^2 = x^2 + y^2$

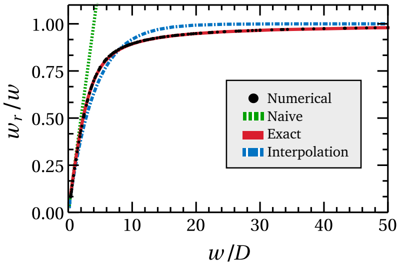
Drift and Diffusion of $\log(r)$

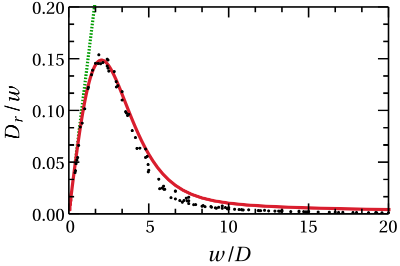
- Initial condition r_0
- $\log(r) \rightarrow$ Normally distributed
- $\log(r) \rightarrow$ Drifts & diffuses
- $\mu = w_r t \quad \sigma^2 = D_r t$
- w_r and D_r - functions of w and D



$r^2 = x^2 + y^2$

Main results: Drift (w_r) and Diffusion (D_r)





- Naive result: $w_r/w = w/4D$ and $D_r/w = w/8D$
- $w_r \;=\; wX_1 \approx w\left[1-\exp\left(-\frac{w}{4D}\right)\right]$
- $D_r \;=\; -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \Delta_n X_n w$

$$X_n \;\equiv\; \langle\cos(2n\varphi)\rangle_\infty \;=\; I_n\left(\frac{w}{2D}\right)/I_0\left(\frac{w}{2D}\right)$$

$$\Delta_n \;\equiv\; C_n(0) \;=\; \frac{1}{2}\left(X_{n+1}+X_{n-1}\right) - X_n X_1$$

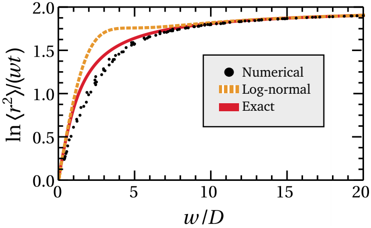
w - Squeeze $\quad D$ - Rotation $\quad I_n(x)$ - Modified Bessel

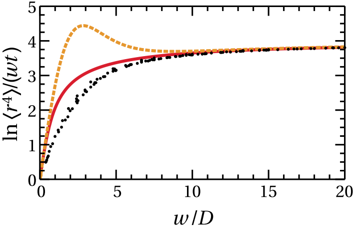
Moments

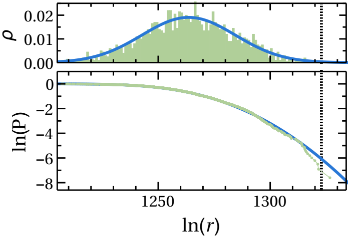
$\log(r) \sim$ Normal: moments of r can be obtained (orange):

$$\log\langle r^n\rangle \;=\; \mu n + \frac{1}{2}\sigma^2 n^2 \;=\; n w_r t + \frac{1}{2} n^2 D_r t$$

Fails when diffusion is big

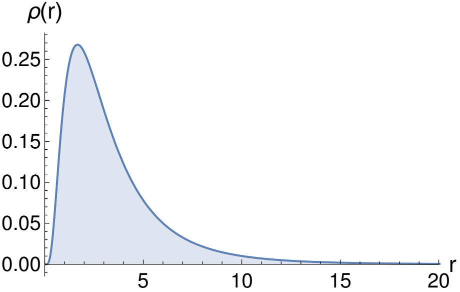


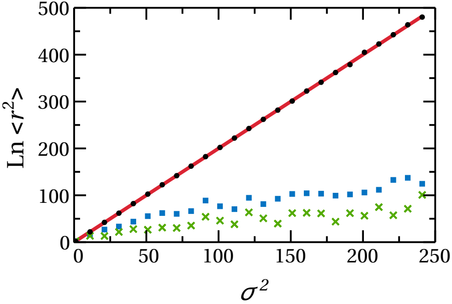




$$\log\langle r^2\rangle/(wt) \sim 2\left[(1+(D/w)^2)^{1/2} - D/w\right]$$

- $\log(r) \sim N(0, \sigma^2)$
- $\log \langle r^2 \rangle = 2\sigma^2$





- Red line - analytical
- Green and Blue, sample mean: 10^2 and 10^5 samples

Part II - Derivations

Drift and Diffusion for the radial coordinates

In polar coordinates $\mathbf{r}$, φ:

$$\begin{aligned}
\dot{\varphi} &= -w\sin(2\varphi) + \Omega(t) \\
\dot{r} &= [w\cos(2\varphi)]\, r \\
\langle \Omega(t)\Omega(t') \rangle &= 2D\delta(t-t')
\end{aligned}$$

$\ln(r)$ performs Brownian motion:

$$\frac{d}{dt}\ln(r) = w\cos(2\varphi)$$

After transient time $\ln(r)$ drifts (w_r) and diffuses (D_r)

Need angular distribution $\rho_(\varphi, t)$

Angular spreading

Langevin:

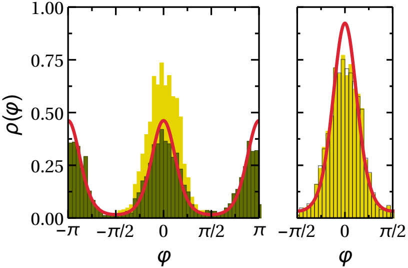
$$\dot{\varphi} \;=\; -w\sin(2\varphi) \;+\; \Omega(t) \qquad\qquad \langle \Omega(t)\Omega(t')\rangle = 2D\delta(t-t')$$

Corresponding Fokker-Planck:

$$\frac{\partial\rho}{\partial t} \;=\; \frac{\partial}{\partial\varphi}\left[\left(D\frac{\partial}{\partial\varphi}+w\sin(2\varphi)\right)\rho\right]$$

Steady state ("detailed balance"):

$$\rho(\varphi) \;\propto\; \exp\left[\frac{w}{2D}\cos(2\varphi)\right]$$



Yellow - $\varphi_0 = 0$
Green - $\varphi_0 = \pi/2$

Angular spreading

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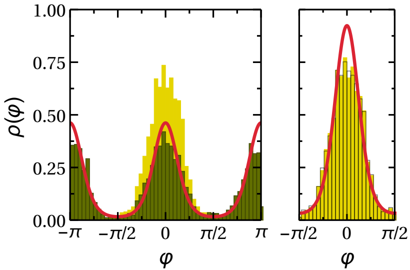

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Drift and Diffusion for the radial coordinates

$\ln(r)$ performs Brownian motion:

$$\frac{d}{dt}\ln(r(t)) \quad = \quad w\cos(2\varphi)$$

Drift:

$$w_r = \lim_{t\to\infty}\frac{d}{dt}\langle\ln(r)\rangle = \langle w\cos(2\varphi)\rangle_{t=\infty} = w\frac{I_1(w/2D)}{I_0(w/2D)}$$

Diffusion:

$$D_r = \frac{1}{2}\lim_{t\to\infty}\frac{d}{dt}\mathrm{Var}\,\ln(r)$$

$$D_r = \int_0^\infty \left[w^2\langle\cos(2\varphi)\cos(2\varphi_t)\rangle_\infty - w_r^2\right]dt$$

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Diffusion coefficient (sketch)

Reminder: $D_r = w^2 \int_0^\infty \left[\langle \cos(2\varphi) \cos(2\varphi_t) \rangle_\infty - \langle \cos(2\varphi) \rangle_\infty^2 \right] dt$

- Use FPE to obtain differential recursive equations involving the integrand.
- Integrate to obtain non-homogenous recursive equation for D_r.
- Solve non-homogeneous equations
  (Use homogeneous solution)
- Obtain D_r:

$$D_r = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \Delta_n X_n w$$

$$X_n \equiv \langle \cos(2n\varphi) \rangle_\infty = I_n\left(\frac{w}{2D}\right) / I_0\left(\frac{w}{2D}\right)$$

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w - Squeeze D - Rotation $I_n(x)$ - Modified Bessel

(even) moments of r

- Find $\langle r^2 \rangle, \langle r^4 \rangle$
- Use **cartesian** coordinates: x, y
- Use FPE to obtain equation of motion for the the moments of x, y

Langevin equation (Stratonovich):

$$\dot{x}_j = v_j + g_j \Omega(t) \qquad \langle \Omega(t)\Omega(t') \rangle = 2D\delta(t - t')$$

with $v_j = (wx, -wy)$ and $g_j = (-y, x)$.

Associated FPE:

$$\frac{d\rho}{dt} = -\frac{\partial}{\partial x_j}\left[v_j\rho - g_j D \frac{\partial}{\partial x_i}(g_i\rho) \right]$$

Moments equation ($X = X(x_i)$):

$$\frac{d}{dt}\langle X \rangle = \left\langle \frac{\partial X}{\partial x_j}\left(v_j + \frac{\partial g_j}{\partial x_i} D g_i \right) \right\rangle$$

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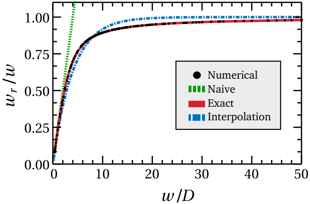
(even) moments of r

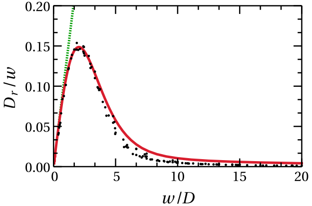
- $X = x^2,\ y^2,\ xy$
- Equations of motion from FPE:

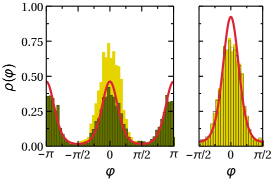
$$\begin{aligned}
\frac{d}{dt}\begin{pmatrix} \langle x^2 \rangle \\ \langle y^2 \rangle \end{pmatrix} &= \left[-2D + 2D\boldsymbol{\sigma}_1 + 2w\boldsymbol{\sigma}_3\right]\begin{pmatrix} \langle x^2 \rangle \\ \langle y^2 \rangle \end{pmatrix} \\
\frac{d}{dt}\langle xy \rangle &= -4D\langle xy \rangle
\end{aligned}$$

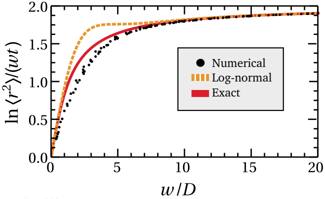
- Initial isotropic distribution, $\langle x^2 \rangle_0 = \langle y^2 \rangle_0 = r_0^2$:

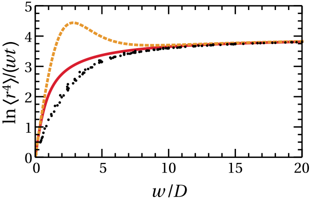
$$\begin{aligned}
\langle r^2 \rangle_t &\approx \frac{r_0^2}{2}\left(1 + \frac{D}{\sqrt{w^2+D^2}}\right) \times \\
&\quad \exp\left[2\left((w^2+D^2)^{1/2} - D\right)t\right]
\end{aligned}$$

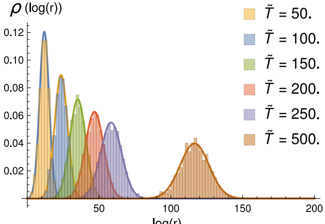

w_r/w
w/D
Numerical
Naive
Exact
Interpolation

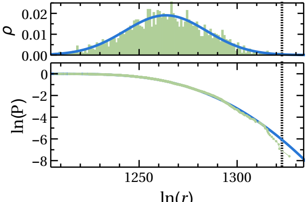

D_r/w
w/D

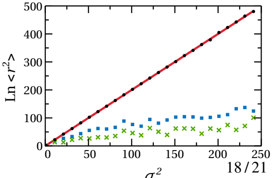

$\rho(\varphi)$
φ
φ


$\ln\langle r^2\rangle/(wt)$
w/D
Numerical
Log-normal
Exact


$\ln\langle r^4\rangle/(wt)$
w/D


ρ (log(r))
log(r)
$\tilde{T}$ = 50.
$\tilde{T}$ = 100.
$\tilde{T}$ = 150.
$\tilde{T}$ = 200.
$\tilde{T}$ = 250.
$\tilde{T}$ = 500.


ρ
ln(P)
$\ln(r)$


Ln <r^2>
σ^2

Extra

Diffusion coefficient - recursive equation

$$D_r = w^2 \int_0^\infty \left[\langle \cos(2\varphi)\cos(2\varphi_t)\rangle_\infty - \langle \cos(2\varphi)\rangle_\infty^2 \right] dt \qquad \text{(Reminder)}$$

Initial delta distribution $\rho_0(\varphi) = \delta(\varphi - \varphi_0)$. Moments:

$$x_n(t;\varphi_0) \equiv \langle \cos(2n\,\varphi_t)\rangle_0 = \langle \cos(2n\,\varphi)\rangle_t = \int \cos(2n\,\varphi)\,\rho(\varphi,t|\varphi_0)d\varphi$$

$$\frac{d}{dt}x_n = -4Dn^2\,x_n + wn\,(x_{n-1} - x_{n+1}) \qquad \text{(FPE and integration by parts)}$$

Steady state $X_n \equiv x_n(\infty)$

$$C_n(t) = \langle x_n(t;\varphi_0)\cos(2\varphi_0)\rangle_\infty - X_n X_1 \qquad \text{(Conditional probability)}$$

$C_1(t)$ is the integrand
$C_n(t)$ obeys the same recusrive equation as x_n

$$c_n \equiv \int_0^\infty C_n(t)dt \qquad \text{(Obtain Recursive equation with } D_r\text{)}$$

$$4Dn^2\,c_n - wn\,(c_{n-1} - c_{n+1}) = \Delta_n \qquad (\mathbf{c_0} = \mathbf{c_\infty} = \mathbf{0})$$

Solve and obtain $D_r = w^2 c_1$

$$D_r = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\Delta_n X_n w \qquad (I_n(x) - \text{Modified Bessel})$$

$$X_n = I_n\left(\frac{w}{2D}\right)/I_0\left(\frac{w}{2D}\right) \qquad \Delta_n \equiv C_n(0) = \frac{1}{2}(X_{n+1} + X_{n-1}) - X_n X_1$$

Solving the recursive equation

Solve:

$$-W_n^+ c_{n+1} + \Lambda_n c_n - W_n^- c_{n-1} = \Delta_n \qquad (\mathbf{c_0} = \mathbf{c_\infty} = \mathbf{0})$$

We have $W_n^+ = -wn \qquad W_n^- = wn \qquad \Lambda_n = 4Dn^2$

First solve homogenous equation. Denote solution X_n:

$$-W_n^+ X_{n+1} + \Lambda_n X_n - W_n^- X_{n-1} = 0 \qquad (X_n \text{ is steady state solution})$$

Otherwise use $R_n = X_n / X_{n-1}$ to reduce to first-order

Rewrite using: $c_n = X_n \tilde{c}_n$

$$-W_n^+ X_{n+1}\tilde{c}_{n+1} + \Lambda_n X_n \tilde{c}_n - W_n^- X_{n-1}\tilde{c}_{n-1} = \Delta_n$$

$$-W_n^+ X_{n+1}(\tilde{c}_{n+1} - \tilde{c}_n) + W_n^- X_{n-1}(\tilde{c}_n - \tilde{c}_{n-1}) = \Delta_n \qquad \text{(Use homogenous solution)}$$

Reduce to first order equation:

$$-W_n^+ X_{n+1}\tilde{a}_{n+1} + W_n^- X_{n-1}\tilde{a}_n = \Delta_n \qquad (\tilde{a}_n = \tilde{c}_n - \tilde{c}_{n-1})$$

$$a_n = \tilde{R}_n \left[\tilde{\Delta}_n + a_{n+1}\right] \qquad \left(\tilde{R}_n = \frac{W_n^+}{W_n^-} R_n,\ \ \tilde{\Delta}_n = \frac{\Delta_n}{W_n^+},\ \ a_n = X_n \tilde{a}_n\right)$$

Solve a_n with $a_\infty = 0$

Finally:

$$c_n = R_n c_{n-1} + a_n \qquad (c_0 = 0)$$

$$c_1 = a_1 = \tilde{R}_1\tilde{\Delta}_1 + \tilde{R}_1\tilde{R}_2\tilde{\Delta}_2 + \ldots = \sum_{n=1}^{\infty} (-1)^n X_n \qquad (\tilde{R}_1 \cdots \tilde{R}_n = (-1)^n X_n)$$