Exercises in Statistical Mechanics

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This exercises pool is intended for a graduate course in "statistical mechanics". Some of the problems are original, while other were assembled from various undocumented sources. In particular some problems originate from exams that were written by B. Horovitz (BGU), S. Fishman (Technion), and D. Cohen (BGU).

[Exercise 9010]

Spin resonance

Spin Resonance: Consider a spin $\frac{1}{2}$ particle with magnetic moment in a constant magnetic field B_0 in the z direction and a perpendicular rotating magnetic field with frequency ω and amplitude B_1; the Hamiltonian is

$$\hat{H} = \hat{H}_0 + \tfrac{1}{2}\hbar\omega_1 \left[\sigma_x cos\left(\omega t\right) + \sigma_y sin\left(\omega t\right)\right]$$

where $\hat{H}_0 = \frac{1}{2}\hbar\omega_0\sigma_z$, $\frac{1}{2}\hbar\omega_0 = \mu B_0$, $\frac{1}{2}\hbar\omega_1 = \mu B_1$ and $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices. The equilibrium density matrix is $\hat{\rho}_{eq} = exp\left(-\beta\hat{H}_0\right)/Tr\left[exp\left(-\beta\hat{H}_0\right)\right]$, so that the heat bath drives the system towards equilibrium with $\hat{H}_0$ while the weak field B_1 opposes this tendency. Assume that the time evolution of the density matrix $\hat{\rho}\left(t\right)$ is determined by

$$d\hat{\rho}/dt = -\frac{i}{h}\left[\hat{H}, \hat{\rho}\right] - \frac{\hat{\rho} - \hat{\rho}_{eq}}{\tau}$$

(a) Show that this equation has a stationary solution of the form $\delta\rho_{11} = -\delta\rho_{22} = a,\ \ \delta\rho_{12} = \delta\rho_{21}^{*} = be^{-i\omega t}$ where $\delta\hat{\rho} = \hat{\rho} - \hat{\rho}_{eq}$.

(b) The term $-\left[\hat{\rho} - \hat{\rho}_{eq}\right]/\tau$ represents $\left(-i/\hbar\right)\left[(\hat{H}_{bath})\hat{\rho}\right]$ where $\hat{H}_{bath}$ is the interaction Hamiltonian with a heat bath. Show that the power absorption is

$$\frac{d}{dt}Tr\left[(\hat{H} + \hat{H}_{bath})\hat{\rho}\right] = Tr\left[\frac{d\hat{H}}{dt}\hat{\rho}\right]$$

(c) Determine b to first order in B_1 (for which $a = 0$ can be assumed), derive the power absorption and show that it has a maximum at $\omega = \omega_0$, i.e. a resonance phenomena. Show that $\left(d/dt\right)Tr\left(\hat{\rho}\hat{H}\right) = 0$, i.e. the absorption is dissipation into the heat bath.