Exercises in Statistical Mechanics

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This exercises pool is intended for a graduate course in "statistical mechanics". Some of the problems are original, while other were assembled from various undocumented sources. In particular some problems originate from exams that were written by B. Horovitz (BGU), S. Fishman (Technion), and D. Cohen (BGU).

====== [Exercise 8490]

Stochastic rate equation

Consider N classical particles in a two site system. The two sites are subjected to a potential difference ε. The temperature of the system is T. Define $n \in [-N, N]$ as the occupation difference. In items (3-6) assume that the thermalization process can be described by a stochastic rate equation

$$\frac{dn}{dt} = -\gamma n + A(t)$$

where $A(t)$ is a noisy term that reflects the fluctuations of the potential difference. Assuming that it has an average value A_0 and a power spectrum $\phi(\omega)$, it follows that n relaxes to an average value $\langle n \rangle$, with fluctuations that are characterized by a power spectrum $C(\omega)$.

(1) Write what is the interaction energy H_{int} of n with the field ε. Later you will have to be careful with the identification of the conjugate variables.

(2) Using the canonical formalism find what are $\langle n \rangle$ and $\text{Var}(n)$. Additionally provide approximations for small ε.

(3) Determined what is A_0 such that $\langle n \rangle$ would be consistent with the canonical result. Assuming small ε deduce that $A_0 \propto \epsilon$, and find the pre-factor.

(4) What is the $\chi(\omega)$ that characterizes the response of n to the applied potential in the linear-response regime? Assume that the dynamics is described by the stochastic rate equation; care to identify correctly the conjugate variables; and take into account your answer to item (3).

(5) Deduce from the fluctuation-dissipation relation what is the power spectrum $C(\omega)$. Care to use the appropriate definition for $\chi(\omega)$, else the result will come out wrong.

(6) Deduce what is the power spectrum $\phi(\omega)$ that is required in order to reproduce $C(\omega)$ from the stochastic rate equation.

Advice: In item (5) verify that your result is consistent with the answer to item (2). Likewise you can debug the numerical pre-factor in your answer to item (6). Care about factors of "2" in your answers. Failure to provide strictly correct pre-factors will be regarded as an essential error.