Ex8484: Galvanometer

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The problem:

A galvanometer can be regarded as a spring-held pointer that has mass M, natural oscillation frequency ω_0, and a damping coefficient γ. The position x of the spring indicates the current I. It obeys the equation

$$\ddot{x} + \omega_0^2 x = -\gamma \dot{x} + A(t) + \alpha I$$

where $A(t)$ represents an environmentally induced white noise that has a spectral intensity ν, and α is a coupling constant.

(1) On the basis of the above Langevin equation write a $d\omega$ integral for the variance $\langle x^2 \rangle$ in the absence of current.

(2) Based on canonical FDT considerations deduce what is the result of the integral that you wrote in the previous item.

(3) For a constant I, what is the average position $\langle x \rangle$ of the pointer?

(4) Regarding I as a driving source, write what is the conjugate variable, what is the interaction term $\mathcal{H}_{int}$ in the Hamiltonian, and what is the associate susceptibility $\chi(\omega)$.

(5) Write an expression for the average rate of energy absorption $\dot{\mathcal{W}}$, given that the current source has a frequency ω and RMS amplitude I_0.

(6) The expression for $\dot{\mathcal{W}}$ is formally the same as for a current source that is connected to a parallel RLC circuit. Write expressions for the effective values of R and L and C.

Tip: The equation of a parallel RLC circuit can be written as $G(\omega)V_\omega = I_\omega$ where $G(\omega)$ is a sum of three terms. Capacitors and inductors are described by $I = C\dot{V}$ and by $V = L\dot{I}$ respectively.

The solution:

(1) In the absence of current the galvanometer equation becomes

$$\ddot{x} + \omega_0^2 x = -\gamma \dot{x} + A(t), \tag{1}$$

taking the Fourier transform we get

$$-\omega^2 x_\omega + \omega_0^2 x_\omega = i\gamma\omega x_\omega + A_\omega. \tag{2}$$

Solving for x_ω we get its response to the white noise A_ω

$$x_\omega = \frac{A_\omega}{\omega_0^2 - \omega^2 - i\gamma\omega}. \tag{3}$$

Using the Wiener-Khinchin theorem

$$\tilde{C}_x(\omega) = \frac{\tilde{C}_A(\omega)}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}. \tag{4}$$

We have defined the correlation function of the stationary stochastic variable as:

$$C(\tau) \equiv C(t_2 - t_1) = \langle A(t_2)A(t_1)\rangle = \nu\delta(t_2 - t_1) \equiv \nu\delta(\tau), \tag{5}$$

thus,

$$\tilde{C}_A(\omega) = \int_{-\infty}^{\infty} C(\tau)e^{i\omega\tau}d\tau = \nu. \tag{6}$$

In that case we know that $\langle x\rangle = 0$ thus,

$$\mathrm{Var}(x) = \langle x^2\rangle = \int_{-\infty}^{\infty} \tilde{C}_x(\omega)d\omega = \int_{-\infty}^{\infty} \frac{\nu}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}d\omega \tag{7}$$

(2) The correlation function $\langle x(t_2)x(t_1)\rangle$ for $t_1 = t_2 = t$ should be consistent with law of equipartition

$$\frac{1}{2}M\omega_0^2\langle x^2\rangle = \frac{T}{2}, \tag{8}$$

thus,

$$\langle x^2\rangle = \frac{T}{M\omega_0^2}. \tag{9}$$

(3) Using the galvanometer equation, the fact that $\langle A(t)\rangle = 0$ and $\langle \ddot{x}\rangle = \langle \dot{x}\rangle = 0$ at equilibrium

$$\langle x\rangle = \frac{I\alpha}{\omega_0^2}. \tag{10}$$

(4) Regarding I as a driving source, we can deduce its interaction term in the Hamiltonian

$$\mathcal{H}_{int} = -\alpha M x I, \tag{11}$$

thus $\alpha M x$ is the conjugate variable of I.
Doing the same procedure as in section (1) for I we get

$$x_\omega = \frac{\alpha I_\omega}{\omega_0^2 - \omega^2 - i\gamma\omega}, \tag{12}$$

but, the generalized force is $\mathcal{F} = -\frac{\partial\mathcal{H}}{\partial I} = \alpha M x$, thus

$$\chi(\omega) = \frac{M\alpha^2}{\omega_0^2 - \omega^2 - i\gamma\omega}. \tag{13}$$

(5) Given $I(t) = Re[I_0 e^{-i\omega t}]$, we know that $\dot{\mathcal{W}} = \eta(\omega)\frac{(I_0\omega)^2}{2}$, where $\eta(\omega) \equiv \frac{\mathrm{Im}[\chi(\omega)]}{\omega}$, thus using (13) we get

$$\dot{\mathcal{W}} = \langle -\dot{I}\alpha M x\rangle = \frac{I_0^2\omega^2}{2}\frac{M\alpha^2\gamma}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}. \tag{14}$$

(6) Since this is a parallel RLC circuit $V_R = V_C = V_L$, thus we can write the equation for the current

$$I_R + I_C + I_L = I(t), \tag{15}$$

where $I(t)$ is the current source.
Using the tip we can rewrite (15)

$$\frac{1}{R}V(t) + C\dot{V} + \int \frac{V(t)}{L}dt = I(t). \tag{16}$$

Taking the Fourier transform we get

$$I_\omega = (\frac{1}{R} - i\omega C - \frac{1}{i\omega L})V_\omega \equiv G(\omega)V_\omega, \tag{17}$$

we can use the relation $I = \dot{Q}$ in order to write

$$V_\omega = \frac{-i\omega}{\frac{1}{R} - i\omega C - \frac{1}{i\omega L}}Q_\omega, \tag{18}$$

thus the susceptibility is

$$\chi(\omega) = \frac{\frac{\omega^2}{C}(\omega^2 - \frac{1}{LC}) - i\frac{\omega^3}{RC^2}}{(\frac{1}{LC} - \omega^2)^2 + \frac{\omega^2}{R^2C^2}}. \tag{19}$$

We can calculate the average rate of energy absorption where V is the generalized force and Q is the parameter

$$\dot{\mathcal{W}} = \langle -\dot{Q}V\rangle = \langle -IV\rangle = \frac{I_0^2}{2}\frac{\frac{\omega^2}{RC^2}}{(\frac{1}{LC} - \omega^2)^2 + \frac{\omega^2}{R^2C^2}}. \tag{20}$$

With analogue to (14) we can find the effective R,L and C

$$L = \frac{M\alpha^2}{\omega_0^2}, \tag{21}$$

$$R = \frac{M\alpha^2}{\gamma}, \tag{22}$$

$$C = \frac{1}{M\alpha^2}. \tag{23}$$