Ex8484: Galvanometer

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The problem:

A galvanometer can be regarded as a spring-held pointer that has mass M, natural oscillation frequency ω_0, and a damping coefficient γ. The position x of the spring indicates the current I. It obeys the equation

$$\ddot{x} + \omega_0^2 x = -\gamma \dot{x} + A(t) + \alpha I$$

where $A(t)$ represents an environmentally induced white noise that has a spectral intensity ν, and α is a coupling constant.

(1) On the basis of the above Langevin equation write a $d\omega$ integral for the variance $\langle x^2 \rangle$ in the absence of current.

(2) Based on canonical FDT considerations deduce what is the result of the integral that you wrote in the previous item.

(3) For a constant I, what is the average position $\langle x \rangle$ of the pointer?

(4) Regarding I as a driving source, write what is the conjugate variable, what is the interaction term $\mathcal{H}_{int}$ in the Hamiltonian, and what is the associate susceptibility $\chi(\omega)$.

(5) Write an expression for the average rate of energy absorption $\dot{W}$, given that the current source has a frequency ω and RMS amplitude I_0.

(6) The expression for $\dot{W}$ is formally the same as for a current source that is connected to a parallel RLC circuit. Write expressions for the effective values of R and L and C.

Tip: The equation of a parallel RLC circuit can be written as $G(\omega)V_\omega = I_\omega$ where $G(\omega)$ is a sum of three terms. Capacitors and inductors are described by $I = C\dot{V}$ and by $V = L\dot{I}$ respectively.

The solution:

(1) In the absence of current the given equation becomes

$$\ddot{x} + \omega_0^2 x = -\gamma \dot{x} + A(t) \tag{1}$$

We take the Fourier transform of this equation and get

$$-\omega^2 x_\omega + \omega_0^2 x_\omega = \gamma i \omega x_\omega + A_\omega \tag{2}$$

which gives the response of x_ω to the white noise A_ω

$$x_\omega = \frac{A_\omega}{\omega_0^2 - \omega^2 - i\omega\gamma} \tag{3}$$

Using the Wiener-Khinchin theorem this becomes

$$\tilde{C}_x(\omega) = \frac{\tilde{C}_A(\omega)}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} \tag{4}$$

$A(t)$ is white noise with spectral intensity ν so its correlation function is

$$C_A(\tau) = \langle A(t_2)A(t_1)\rangle = \nu\delta(t_2 - t_1) \tag{5}$$

Meaning its power spectrum is

$$\tilde{C}_A(\omega) = \int_{-\infty}^{\infty} dt' \left\langle A(t')A(0)\right\rangle e^{i\omega t'} = \nu \tag{6}$$

plugging (6) into (4) and taking the inverse Fourier transform with $t = 0$ we get

$$\left\langle x^2\right\rangle = C_x(t=0) = \int_{-\infty}^{\infty} \tilde{C}_x(\omega)d\omega = \int_{-\infty}^{\infty} \frac{\nu}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}d\omega \tag{7}$$

(2) Based on canonical considerations, we simply use the law of equipartition

$$\frac{1}{2}m\omega_0^2 \left\langle x^2\right\rangle = \frac{T}{2} \tag{8}$$

to get

$$\left\langle x^2\right\rangle = \frac{T}{m\omega_0^2} \tag{9}$$

(3) At equilibrium $\langle\ddot{x}\rangle = \langle\dot{x}\rangle = 0$, and $\langle A\rangle = 0$ by definition, so $\langle x\rangle$ can be extracted from the Langevin equation

$$\langle x\rangle = \frac{\alpha I}{\omega_0^2} \tag{10}$$

(4) From the Langevin equation we can deduce the interaction term to be

$$\mathcal{H}_{int} = -\alpha m x I \tag{11}$$

meaning $\alpha m x$ is the conjugate variable of I. Repeating the procedure done in (1) with $I \neq 0$, we get

$$\langle x\rangle_\omega = \frac{\alpha I_\omega}{\omega_0^2 - \omega^2 - i\omega\gamma} \tag{12}$$

which implies

$$\chi(\omega) = \frac{m\alpha^2}{\omega_0^2 - \omega^2 - i\omega\gamma} \tag{13}$$

(5) For $I(t) = I_0\cos(\omega t)$, the average rate of energy absorption $\dot{W}$ is given by

$$\dot{W} = \left\langle -\dot{X}\mathcal{F}\right\rangle_t = \left\langle -\dot{I}\alpha m x\right\rangle_t = \frac{I_0^2\omega^2}{2}\eta(\omega) = \frac{I_0^2\omega}{2}\mathrm{Im}[\chi(\omega)] \tag{14}$$

and plugging (13) in we get

$$\dot{W} = \frac{I_0^2}{2}\frac{m\alpha^2\omega^2\gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} \tag{15}$$

(6) For each of the components in the parallel RLC circuit we have

$$V_L = L\dot{I}_L \implies I_L(\omega) = \frac{V_L(\omega)}{-i\omega L} \tag{16a}$$

$$V_R = I_R R \implies I_R(\omega) = \frac{V_R(\omega)}{R} \tag{16b}$$

$$V_C = \frac{Q_C}{C} \implies I_C(\omega) = -i\omega C V_C(\omega) \tag{16c}$$

The sum of all currents must equal that of the source, and since all components are connected in parallel, $V_L = V_C = V_R$, leading to

$$I_\omega = \left(-\frac{1}{i\omega L} + \frac{1}{R} - i\omega C\right) V_\omega \tag{17}$$

Using the relation between charge and current, $\dot{Q} = I$, we rewrite this equation as

$$V_\omega = \frac{-i\omega}{-\frac{1}{i\omega L} + \frac{1}{R} - i\omega C} Q_\omega \tag{18}$$

where V is the generalized force and Q the parameter. Hence $\chi(\omega)$ is

$$\chi(\omega) = \frac{\frac{\omega^2}{C}\left(\omega^2 - \frac{1}{LC}\right) - i\frac{\omega^3}{RC^2}}{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{\omega^2}{R^2C^2}} \tag{19}$$

The energy absorption is then given by

$$\dot{W} = \left\langle -\dot{Q}V \right\rangle = \langle -IV \rangle = \frac{I_0^2}{2\omega}\mathrm{Im}[\chi(\omega)] = \frac{I_0^2}{2} \frac{\frac{\omega^2}{RC^2}}{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{\omega^2}{R^2C^2}} \tag{20}$$

and by comparison with (15) we get the following effective values

$$L = \frac{m\alpha^2}{\omega_0^2} \tag{21a}$$

$$R = \frac{m\alpha^2}{\gamma} \tag{21b}$$

$$C = \frac{1}{m\alpha^2} \tag{21c}$$