Galvanometer
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Exercise 8484

A galvanometer at temperature T has a mass m, a deflection spring with oscillation frequency ω_0 and a damping resistance R. The position x of the spring measures the current I via the equation

$$\ddot{x} + \omega_0^2 x = -\gamma \dot{x} + A(t) + \alpha I$$

where $A(t)$ is a random force, γ is the friction and α is an instrument parameter, converting the current into force.

a. Find the current condition for recording?

b. Evaluate dissipation rate for current I with frequency ω.

c. Identify R by equating b. with $I^2 R$.

d. Rewrite a. in terms of R in the limit $\omega \sim \omega_0$.

Solution

a.

$$\ddot{x} + \omega_0^2 x = -\gamma\dot{x} + A(t) + \alpha I \tag{1}$$

In equilibrium $\ddot{x} = \dot{x} = 0$, and for the expectation value $< x >$ the random force $A(t)$ disappears and we get

$$< x >= \frac{\alpha I}{\omega_0^2} \tag{2}$$

Now we will define $\tilde{x} = x - < x >$ and use the equipartition law

$$\frac{1}{2} m\omega_0^2 < \tilde{x}^2 >= \frac{1}{2} k_b T \tag{3}$$

for accurate measurement will demand

$$< x^2 > > < \tilde{x}^2 > \tag{4}$$

by using condition (4) in equation (3) we will get the recording condition for I.

$$\left(\frac{\alpha I}{\omega_0^2}\right)^2 > \left(\frac{k_b T}{m\omega_0^2}\right) \Rightarrow I^2 > \frac{k_b T \omega_0^2}{m\alpha^2} \tag{5}$$

b. The driving force is

$$f_0 = m\alpha I \tag{6}$$

and the dissipation is

$$\frac{d\overline{E}}{dt} = \frac{1}{2}\omega |f_0|^2 Im(\alpha_x(\omega)) \tag{7}$$

we shall use langevin equation in fourier space

$$(-m\omega^2 - im\gamma\omega + m\omega_0^2)x_\omega = mA(\omega) + f(\omega) \tag{8}$$

and again for the expection value A will fall.

$$< x_\omega >= \alpha_x(\omega) f(\omega) \tag{9}$$

by using equation (8) and (9), and rearrange it we will get

$$\alpha_x(\omega) = (-m\omega^2 - im\gamma\omega + m\omega_0^2)^{-1} = \frac{1}{m(\omega_0^2 - \omega^2 - i\gamma\omega)} \tag{10}$$

from (10) will get $Im(\alpha_x(\omega))$

$$Im(\alpha_x(\omega)) = \frac{\gamma\omega/m}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \tag{11}$$

Placing (11) in (7) will get

$$\frac{d\overline{E}}{dt} = \frac{1}{2}\frac{m\alpha^2\gamma\omega^2 I_0^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \tag{12}$$

we used the fact that the only frequency that lasts is the current frequency and that the average mean square of the current is

$$I = I_0 \cos(\omega t) \Rightarrow \overline{I^2} = \frac{I_0^2}{2} \tag{13}$$

c.

$$\frac{d\overline{E}}{dt} = R\overline{I^2} \Rightarrow \frac{RI_0^2}{2} = \frac{I_0^2}{2}\frac{m\alpha^2\gamma\omega^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} \tag{14}$$

$$R = \frac{m\alpha^2\gamma\omega^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} \tag{15}$$

d. Equation (15) in the limit $\omega \sim \omega_0$ becomes

$$R \underset{\omega\to\omega_0}{=} \frac{\alpha^2 m}{\gamma} \tag{16}$$

we get the current condition in R terms by placing (16) in (5)

$$I^2 > \frac{k_B T\omega_0^2}{\gamma R} \tag{17}$$