Ex8483: Millikan experiment

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The problem:

Consider a Millikan-type experiment whose purpose is to measure the charge e of a particle with mass m. The particle is located beteen plates of capacitor, where the electric field $\mathcal{E}$ is in the "up" direction, while the gravitation g is in the "down" direction. The distance between the plates is L, and the temperature of the system is T. Due to the poor vacuum the particle executes a Brownian motion that is described by a Langevin equation with friction force $-\eta v$. The charge of the electron is estimated via $\delta F = e\mathcal{E} - \mathsf{m}g = 0$. In item (1) the system is prepared with a single particle in the middle. In item (3) assume a uniform gas of N particles. In both cases the current is integrated during a time interval t, and the charge $Q = \int I(t')dt'$ is inspected as "readout".

(1) Assuming that $\delta F = 0$, determine the time t_d such that for $t \ll t_d$ it is not likely to get charge readout.

(2) What is the δF for which the condition $t \ll t_d$ is no longer valid. We shall regard this value, call it δ_1, as the resolution of the measurement.

(3) Assuming that $\delta F = 0$, determine the power spectrum $C(\omega)$ of the current $I(t)$.

(4) Assume that the time of the measurement is t. What is the δF for which the condition $\sqrt{\text{var}(Q)} \ll \langle Q \rangle$ is no longer valid. We shall regard this value, call it δ_N, as the resolution of the measurement.

(5) Express the ratio δ_N/δ_1 as a function of N and t/t_d.

Tips: In the absence of fluctuations $\delta F = 0$ is indicated by having zero readout. In item (3) the "readout" is a current versus voltage ("IV") measurement, and $\delta F = 0$ is indicated by zero current. Due to the fluctuations there is some blurring which determines the resolution δ_N. In order to calculate the fluctuations in item (3) define the one-particle current as the velocity (up to a prefactor).

The solution:

(1) The Langevin equation is a stochastic force equation describing the motion of a particle executing Brownian motion:

$$\dot{v} = -\eta v + f(t) \tag{1}$$

Where η encompasses all drag forces acting on the particle (in this case collisions) and $f(t)$ is a random force with $< f(t) >= 0$. Solving this equation for the spreading of the particle yields $\left\langle (x(0) - x(t))^2 \right\rangle = 2Dt$ and, for timescales longer than decorrelation time, D is constant and given by the Einstein relation $D = \frac{T}{\eta}$. It follows, that it would be unlikely to get a charge readout for:

$$\frac{T}{\eta} \cdot t \ll L^2 \longrightarrow t_d = \frac{\eta L^2}{T} \tag{2}$$

(2) When $e\mathcal{E} - mg \neq 0$ a "drift" term is to be added to the Langevin equation:

$$\dot{v} = -\eta v + f(t) + \delta F \tag{3}$$

Where $\delta F = e\mathcal{E} - mg$. The average velocity is no longer zero, and is given by $< v >= \frac{\delta F}{\eta}$. In this case a minimum measurment time $t > \frac{L}{<v>}$ is required to get a reading. But we would also want this time to be shorter than the spreading time t_d we found in the previous item. This leads to the condition:

$$\delta F > \frac{T}{L} \equiv \delta_1 \tag{4}$$

(3) The current of a single particle is $I^1 = \frac{e}{L}v$. The power spectrum can be expressed as:

$$< |I^1_\omega|^2 >= \left(\frac{e}{L}\right)^2 < |v_\omega|^2 > \tag{5}$$

In frequency space, Eq(1) can be rewritten as:

$$v_\omega = (-i\omega + \gamma)^{-1} \frac{f_\omega}{m} \; \left(\gamma = \frac{\eta}{m}\right) \tag{6}$$

From which the velocity power spectrum can be easily found:

$$< |v_\omega|^2 >= (\omega^2 + \gamma^2)^{-1} \frac{C_\omega}{m^2} \tag{7}$$

C_ω in the equation above is the, ensemble averaged, power of f (which is also the correlation function). For white noise, the Einstein relation gives us $C_\omega = 2m\gamma T$ (refered to as ν in D.C. notes). The total current is a sum over single particle currents and so the power of the total current will be N times the power from a single particle:

$$< |I_\omega|^2 >= N \left(\frac{e}{L}\right)^2 \frac{T}{m} \frac{2\gamma}{\omega^2 + \gamma^2} \tag{8}$$

(4) The readout is the total charge $Q = \int_0^t I(t')dt'$. For a significant readout we require $\sqrt{\text{var}(Q)} \ll \langle Q \rangle$. δF contributes to the RHS:

$$< Q >=< I > t = N\frac{e}{L} < v >= N\frac{e}{L}\frac{\delta F}{\eta}t \tag{9}$$

and the current variance contributes to the LHS (it can be calculated assuming $\delta F = 0$):

$$Var(Q) =< Q^2 >= N \left(\frac{e}{L}\right)^2 \int_0^t \int_0^t dt'dt'' < v(t')v(t'') >= N \left(\frac{e}{L}\right)^2 \frac{2T}{\eta} t \tag{10}$$

The condition on δF is then:

$$\delta F > \sqrt{\frac{T\eta}{Nt}} \equiv \delta_N \tag{11}$$

(5) The ratio $\frac{\delta_N}{\delta_1}$ can be expressed as a function of N and t/t_d:

$$\frac{\delta_N}{\delta_1} = \frac{1}{\sqrt{N\frac{t}{t_d}}} \tag{12}$$