Ex8481: Mass on a spring

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The problem:

A balance for measuring weight consists of a sensitive spring which hangs from a fixed point. The spring constant is K. The balance is at temperature T and gravity acceleration is g in the x direction. A small mass m hangs at the end of the spring. There is an option to apply an external force $F(t)$, to which x is conjugate or apply an external vector potential $A(t)$.

(a) Find the partition function Z.

(b) Find $\langle x \rangle$ and $\langle x^2 \rangle$ and $\text{Var}(x)$.

(c) Write a Langevin equation for $x(t)$, with friction γ, and a random force $f(t)$.

(d) Assuming $\langle f(t)f(0) \rangle = C\delta(t)$, find $\text{Var}(x)$, and deduce what is C by comparing with the canonical result.

(e) Assuming x is measured in the lab by averaging over time period t_0, what is the minimal mass that can be meaningfully measured?

(f) Describe the external force $F(t)$ by a scalar potential and demonstrate FDT.

(g) Describe the external force $F(t)$ by a vector potential and demonstrate FDT.

Note: $\int \frac{d\omega}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2} = \frac{\pi}{\gamma\omega_0^2}$.

The solution:

(a) The Hamiltonian of the system is:

$$H = \frac{p^2}{2m} + \frac{1}{2}Kx^2 - mgx \tag{1}$$

We can rewrite the Hamiltonian in the following way

$$H = \frac{p^2}{2m} + \frac{1}{2}K(x - x_0)^2 + const \tag{2}$$

where $x_0 = \frac{mg}{K}$. Therefore, the partition function is

$$Z = \frac{1}{\lambda_T}\int_{-\infty}^{\infty} e^{-\beta\frac{K}{2}(x-x_0)^2}dx \times const = \frac{1}{\lambda_T}\sqrt{\frac{2\pi}{\beta K}} \times const \tag{3}$$

(b) The Gaussian in the partition function is centered around x_0, therefore we deduce that

$$\langle x \rangle = x_0 \tag{4}$$

In order to find $\langle x^2 \rangle$ we use the equipartition and the Virial theorems - $\langle x\frac{\partial H}{\partial x} \rangle = T$,

$$\langle xK(x - x_0) \rangle = T \tag{5}$$

$$\langle x^2 \rangle = \frac{T}{K} + x_0^2 \tag{6}$$

And the variance of x is

$$\mathrm{Var}(x) = \langle x^2 \rangle - \langle x \rangle^2 = \frac{T}{K} \tag{7}$$

(c) The Langevin equation for this system is:

$$m\ddot{x} + \gamma\dot{x} + Kx - mg = f(t) \tag{8}$$

(d) Redefining x around the equilibrium, $x \to x - x_0$, so that in equilibrium $\langle x \rangle = 0$, Eq. (8) becomes

$$m\ddot{x} + \gamma\dot{x} + Kx = f(t) \tag{9}$$

Fourier transforming Eq. (9)

$$(-m\omega^2 - i\gamma\omega + K)x_\omega = f_\omega \tag{10}$$

Using the Wiener-Khinchin theorem, we get

$$\tilde{C}_{xx}(\omega) = \frac{\tilde{C}_{ff}(\omega)}{(-m\omega^2 + K)^2 + (\gamma\omega)^2} \tag{11}$$

where the Fourier transform of the force-correlation is $\tilde{C}_{ff}(\omega) = C$.
Using Eq. (11) we calculate $\mathrm{Var}(x)$ -

$$\mathrm{Var}(x) = C_{xx}(t=0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{C}_{xx}(\omega)d\omega = \frac{1}{2\pi}\frac{C}{m^2}\int_{-\infty}^{\infty} \frac{d\omega}{(\omega^2 - \frac{K}{m})^2 + (\frac{\gamma\omega}{m})^2} = \frac{C}{2\gamma K} \tag{12}$$

Finally, comparing the result with with Eq (7), we get

$$C = 2\gamma T \tag{13}$$

Hence, Eq. (11) becomes

$$\tilde{C}_{xx}(\omega) = \frac{2\gamma T}{m^2[(\omega^2 - \frac{K}{m})^2 + (\frac{\gamma\omega}{m})^2]} \tag{14}$$

(e) We measure $x(t)$ in the lab and average over time period t_0. This measurement introduces a new random variable X -

$$X = \frac{1}{t_0}\int_0^{t_0} x(t)dt \tag{15}$$

This variable has a mean value $\langle X \rangle$ and standard deviation $\sigma(X)$. In order to get a meaningful measurement, it has to obey the condition - $\langle X \rangle \gg \sigma(X)$. From this condition, the minimal mass $m_{\min}$ will be found. The mean value of X is

$$\langle X \rangle = x_0 \tag{16}$$

In order to find $\sigma(X)$ we subtract the mean value from X

$$X - x_0 = \frac{1}{t_0}\int_0^{t_0} \big(x(t) - x_0\big)dt \tag{17}$$

Redefining $x \to x - x_0$ (so that $\langle x \rangle = 0$) yields

$$\sigma^2(X) = \frac{1}{t_0^2}\int_0^{t_0} dt' \int_0^{t_0} dt'' \langle x(t')x(t'') \rangle = \frac{1}{t_0}\tilde{C}_{xx}(\omega = 0) = \frac{2\gamma T}{t_0 K^2} \tag{18}$$

where we used $C_{xx}(t'-t'') = \langle x(t')x(t'')\rangle$.
Finally, the minimal mass is given by

$$\langle X\rangle \gg \sigma(X) \tag{19}$$

$$m \gg \frac{1}{g}\sqrt{\frac{2\gamma T}{t_0}} = m_{\min} \tag{20}$$

(f) The force $F(t)$ is described by a scalar potential ε, so the conjugate variables are x and ε. Averaging the Langevin formula

$$m\langle\ddot{x}\rangle + \gamma\langle\dot{x}\rangle + K\langle x\rangle = \varepsilon \tag{21}$$

and Fourier transforming Eq. (21), we get

$$x_\omega = \chi_\omega \varepsilon_\omega \tag{22}$$

where $\chi_\omega = \frac{1}{(-m\omega^2+K)-i\omega\gamma}$. Hence, we get the correlation function

$$\tilde{C}_{xx}(\omega) = \frac{2T}{\omega}\,\mathrm{Im}\{\chi_\omega\} = \frac{2\gamma T}{m^2[(\omega^2-\frac{K}{m})^2+(\frac{\gamma\omega}{m})^2]} \tag{23}$$

This result is exactly the same as Eq. (14).

(g) The force $F(t)$ is described by a vector potential $A(t)$, so the conjugate variables are v and A. The averaged Langevin formula becomes

$$m\langle\dot{v}\rangle + \gamma\langle v\rangle + K\langle x\rangle = -\dot{A} \tag{24}$$

The correlation function is

$$\tilde{C}_{vv}(\omega) = \omega^2\tilde{C}_{xx}(\omega) = \frac{2\gamma T\omega^2}{m^2[(\omega^2-\frac{K}{m})^2+(\frac{\gamma\omega}{m})^2]} \tag{25}$$

Next, we calculate $\mathrm{Var}(v)$

$$\mathrm{Var}(v) = C_{vv}(t=0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{C}_{vv}(\omega)d\omega = \frac{T}{m} \tag{26}$$

where we have used the identity $\int\limits_{-\infty}^{\infty} \frac{x^2}{(x^2-a^2)^2+(bx)^2}dx = \frac{\pi}{b}$.

This result coincides with the equipartition theorem - $\frac{m\langle v^2\rangle}{2} = \frac{T}{2}$.