

Ex8481: Mass on a spring

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The problem:

A balance for measuring weight consists of a sensitive spring which hangs from a fixed point. The spring constant is K . The balance is at temperature T and gravity acceleration is g in the x direction. A small mass m hangs at the end of the spring. There is an option to apply an external force $F(t)$, to which x is conjugate or apply an external vector potential $A(t)$.

- Find the partition function Z .
- Find $\langle x \rangle$ and $\langle x^2 \rangle$ and $\text{Var}(x)$.
- Write a Langevin equation for $x(t)$, with friction γ , and a random force $f(t)$.
- Assuming $\langle f(t)f(0) \rangle = C\delta(t)$, find $\text{Var}(x)$, and deduce what is C by comparing with the canonical result.
- Assuming x is measured in the lab by averaging over time period t_0 , what is the minimal mass that can be meaningfully measured?
- Describe the external force $F(t)$ by a scalar potential and demonstrate FDT.
- Describe the external force $F(t)$ by a vector potential and demonstrate FDT.

Note: $\int \frac{d\omega}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2} = \frac{\pi}{\gamma \omega_0^2}$.

The solution:

- The Hamiltonian of the system is:

$$H = \frac{p^2}{2m} + \frac{1}{2}Kx^2 - mgx \quad (1)$$

We rewrite the Hamiltonian in the following way

$$H = \frac{p^2}{2m} + \frac{1}{2}K(x - x_0)^2 + \text{const} \quad (2)$$

where $x_0 = \frac{mg}{K}$. Therefore, the partition function is

$$Z = \frac{1}{\lambda_T} \int_{-\infty}^{\infty} e^{-\beta \frac{K}{2}(x-x_0)^2} \times \text{const} \quad (3)$$

- The Gaussian in the partition function is centered around x_0 , therefore we deduce that

$$\langle x \rangle = x_0 \quad (4)$$

In order to find $\text{Var}(x)$ we use the equipartition theorem - $\langle x \frac{\partial H}{\partial x} \rangle = T$,

$$\langle xK(x - x_0) \rangle = T \quad (5)$$

$$\langle x^2 \rangle = \frac{T}{K} + x_0^2 \quad (6)$$

The variance of x is

$$\text{Var}(x) = \langle x^2 \rangle - \langle x \rangle^2 = \frac{T}{K} \quad (7)$$

(c) The Langevin equation for this system is:

$$m\ddot{x} + \gamma\dot{x} + Kx - mg = f(t) \quad (8)$$

(d) Redefining x around the equilibrium, $x \rightarrow x - x_0$, so that in equilibrium $\langle x \rangle = 0$, Eq. (8) becomes

$$m\ddot{x} + \gamma\dot{x} + Kx = f(t) \quad (9)$$

Fourier transforming Eq. (9)

$$(-m\omega^2 - i\gamma\omega + K)x_\omega = f_\omega \quad (10)$$

Using the Weiner-Khinchin theorem, we get

$$\tilde{C}_{xx}(\omega) = \frac{\tilde{C}_{ff}(\omega)}{(-\omega^2 m + K)^2 + (\gamma\omega)^2} \quad (11)$$

where the Fourier transform of the force-correlation is $\tilde{C}_{ff}(\omega) = C$.

Using Eq. (11) we calculate $\text{Var}(x)$ -

$$\text{Var}(x) = C_{xx}(t=0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{C}_{xx}(\omega) d\omega = \frac{1}{2\pi} \frac{C}{m^2} \int_{-\infty}^{\infty} \frac{d\omega}{(\omega^2 - \frac{K}{m})^2 + (\frac{\gamma\omega}{m})^2} = \frac{C}{2\gamma K} \quad (12)$$

Finally, comparing the result with with Eq (7), we get

$$C = 2\gamma T \quad (13)$$

Hence, Eq. (11) becomes

$$\tilde{C}_{xx}(\omega) = \frac{2\gamma T}{m^2 [(\omega^2 - \frac{K}{m})^2 + (\frac{\gamma\omega}{m})^2]} \quad (14)$$

(e) We measure $x(t)$ in the lab and average over time period t_0 . This measurement introduces a new random variable X -

$$X = \frac{1}{t_0} \int_0^{t_0} x(t) dt \quad (15)$$

This variable has a mean value $\langle X \rangle$ and standard deviation $\sigma(X)$. In order that the measured value of X will be within the resolution of the measurement, it has to obey the condition - $\langle X \rangle \gg \sigma(X)$. From this condition, the minimal mass m_{\min} will be found. The mean value of X is

$$\langle X \rangle = x_0 \quad (16)$$

In order to find $\sigma(X)$ we subtract the mean value from X

$$X - x_0 = \frac{1}{t_0} \int_0^{t_0} (x(t) - x_0) dt \quad (17)$$

Redefining $x \rightarrow x - x_0$ yields

$$\sigma^2(X) = \frac{1}{t_0^2} \int_0^{t_0} dt' \int_0^{t_0} dt'' \langle x(t')x(t'') \rangle = \frac{1}{2t_0} \tilde{C}_{xx}(\omega=0) = \frac{T\gamma}{t_0 K^2} \quad (18)$$

where we used $C_{xx}(t' - t'') = \langle x(t')x(t'') \rangle$.
 Finally, the minimal mass is given by

$$\langle X \rangle \gg \sigma(X) \quad (19)$$

$$m \gg \frac{1}{g} \sqrt{\frac{T\gamma}{t_0}} = m_{\min} \quad (20)$$

(f) The force $F(t)$ is described by a scalar potential ε , so the conjugate variables are x and ε .
 Averaging the Langevin formula

$$m\langle \ddot{x} \rangle + \gamma\langle \dot{x} \rangle + K\langle x \rangle = \varepsilon \quad (21)$$

and Fourier transforming Eq. (22), we get

$$x_\omega = \chi_\omega \varepsilon_\omega \quad (22)$$

where $\chi_\omega = \frac{1}{(-m\omega^2 + K) - i\omega\gamma}$. Hence, we get the correlation function

$$\tilde{C}_{xx}(\omega) = \frac{2T}{\omega} \text{Im}\{\chi_\omega\} = \frac{2\gamma T}{m^2[(\omega^2 - \frac{K}{m})^2 + (\frac{\gamma\omega}{m})^2]} \quad (23)$$

This result is exactly the same as Eq. (14).

(g) The force $F(t)$ is described by a vector potential $A(t)$, so the conjugate variables are v and A .
 The averaged Langevin formula becomes

$$m\langle \dot{v} \rangle + \gamma\langle v \rangle + K\langle x \rangle = -\dot{A} \quad (24)$$

Fourier transforming Eq. (26), we get

$$v_\omega = \chi_\omega A_\omega \quad (25)$$

where $\chi_\omega = \frac{\omega^2}{-m\omega^2 - i\gamma\omega + K}$. The correlation function is

$$\tilde{C}_{vv}(\omega) = \frac{2T}{\omega} \text{Im}\{\chi_\omega\} = \frac{2\gamma T \omega^2}{(K - m\omega^2)^2 + (\gamma\omega)^2} \quad (26)$$

Next, we calculate $\text{Var}(v)$

$$\text{Var}(v) = C_{vv}(t=0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{C}_{vv}(\omega) d\omega = \frac{T}{m} \quad (27)$$

where we have used the identity $\int_{-\infty}^{\infty} \frac{x^2}{(x^2 - a^2)^2 + (bx)^2} dx = \frac{\pi}{b}$. This result coincides with the equipartition theorem - $\frac{m\langle v^2 \rangle}{2} = \frac{T}{2}$.