E8481: Mass on a spring

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The problem:

A balance for measuring weight consists of a sensitive spring which hangs from a fixed point. The spring constant is K. The balance is at temperature T and gravity acceleration is g in x direction. A small mass m hangs at the end of the spring. There is an option to apply an external force $F(t)$, to which x is conjugate or apply an external Vector potential $A(t)$.

(a) Find the partition function Z.

(b) Find the average $\langle x(t)\rangle$ and $\langle x^2(t)\rangle$.

(c) Find the fluctuation $\langle \delta x^2\rangle = \langle (x-\langle x\rangle)^2\rangle$, what is the minimal m which can be meaningfully measured?

(d) Write the Langevin equation for $x(t)$ with friction γ and a random force $f(t)$.

(e) Assuming $\langle f(t)f(0)\rangle = C\delta(t)$, Find $\langle \tilde{x}^2(t)\rangle$ and the intensity of the random force $f(t)$ that acts on the mass , from (b) find the coefficient C.

(f) Describe the external force by a scalar potential and demonstrate FDT.

(g) Describe the external force by a vector potential and demonstrate FDT.

The solution:

(a) The Hamiltonian of the system is:

$$H = \frac{p^2}{2m} + \frac{1}{2}Kx^2 - mgx$$

$$Z = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{-\frac{\beta p^2}{2m}}\,dp \int_{-\infty}^{+\infty} e^{-(\frac{\beta K x^2}{2}-mg\beta x)}\,dx = \frac{1}{2\pi\hbar}\sqrt{\frac{2m\pi}{\beta}}\int_{-\infty}^{+\infty} e^{-(y^2-\sqrt{\frac{2\beta}{K}}mgy)}\,dy = \{y=\sqrt{\frac{\beta K}{2}}x\}$$

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$$= \frac{1}{\hbar\beta}\sqrt{\frac{m}{K\pi}}\int_{-\infty}^{+\infty} e^{-(y-\sqrt{\frac{\beta}{2K}}mg)^2}\,e^{\frac{\beta m^2 g^2}{2K}}\,dy = \frac{1}{\hbar\beta}\sqrt{\frac{m}{K}}\,e^{\frac{\beta m^2 g^2}{2K}}$$

(b) By definition of $\langle x\rangle$

$$\langle x(t)\rangle = \frac{1}{Z}\frac{1}{2\pi\hbar}\int\limits_{-\infty}^{+\infty} e^{-\frac{\beta p^2}{2m}}\,dp\int\limits_{-\infty}^{+\infty} x\,e^{-(\frac{\beta K x^2}{2}-mg\beta x)}\,dx$$

$$= \sqrt{\frac{\beta K}{2\pi}}\,e^{-\frac{\beta m^2 g^2}{2K}}\{-\frac{1}{\beta K}e^{-(\frac{\beta K x^2}{2}-mg\beta x)}\Big|_{-\infty}^{+\infty} + \frac{mg}{K}\int\limits_{-\infty}^{+\infty} e^{-(\frac{\beta K x^2}{2}-mg\beta x)}dx\}$$

The first part of the integral goes to zero and we get

$$\langle x\rangle = \frac{mg}{k}$$

From the virial theorem$\langle x\frac{\partial H}{\partial x}\rangle = k_B T$

$$\langle x(Kx-mg)\rangle = K\langle x^2\rangle - mg\langle x\rangle = K\langle x^2\rangle - mg(\frac{mg}{K}) = k_B T$$

$$\langle x^2\rangle = \frac{k_B T}{K} + (\frac{mg}{K})^2$$

(c) $\langle \delta x^2\rangle = \langle (x-\langle x\rangle)^2\rangle = \langle x^2\rangle - \langle x^2\rangle = \dfrac{k_B T}{K}$

The Fluctuation of the spring should be much smaller than the tension itself, Therefor

$$\langle x\rangle^2 \gg \langle \delta x^2\rangle \quad\rightarrow\quad \frac{k_B T}{K} \ll \left(\frac{mg}{k}\right)^2$$

And, finally, we get following condition: $m \gg \sqrt{\frac{K k_B T}{g^2}}$

(d) We use Hamilton equations:

1. $\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x}$
2. $\frac{\partial H}{\partial x} = Kx - mg = -\dot{p} = -m\ddot{x}$

from the two equations, friction γ that interact with velocity and a random force $A(t)$ we find a Langevin's equation (the system is in equilibrium)

$$m\ddot{x} + m\gamma\dot{x} + Kx - mg = mf(t)$$

(e) By set $\tilde{x}(t) = x(t) + \langle x(t)\rangle$into the Langevin's equation that we got and using Fourier transform functional relationships $\frac{\partial^n}{\partial t^n}x(t) = (i\omega)^n X(\omega)$ we get

$(-m\omega^2 + i\omega m\gamma + K)\tilde{X}(\omega) = mf(\omega)$ we rearrange a little to have

$$\frac{|\tilde{X}(\omega)|^2}{|f(\omega)|^2} = \frac{1}{\omega^2}\frac{|\tilde{V}(\omega)|^2}{|f(\omega)|^2} = \frac{m^2}{(m\omega^2 - k)^2 + m^2\omega^2\gamma^2}$$

The intensity of the random force $A(t)$ that acts on the mass

$$\nu_A(\omega) = \int_{-\infty}^{+\infty} C_A(t)\, e^{i\omega t} dt = \int_{-\infty}^{+\infty} C\delta(t) e^{i\omega t} dt = C \;\; where \;\; C_A(t) = \langle f(t) f(0) \rangle$$

From the following relation $\frac{\nu_x(\omega)}{\nu_A(\omega)} = \frac{|\tilde{X}(\omega)|^2}{|f(\omega)|^2} \;\rightarrow\; \nu_x(\omega) = \frac{C}{(\omega^2 - k/m)^2 + \omega^2\gamma^2}$

$$\langle \tilde{x}(t)\tilde{x}(t+\tau) \rangle = \int_{-\infty}^{+\infty} \nu_x(\omega) e^{-i\omega\tau} d\omega \;\;\rightarrow\;\; \langle \tilde{x}^2(t) \rangle = \int_{-\infty}^{+\infty} \nu_x(\omega) d\omega = \frac{1}{2\pi}\int \frac{d\omega}{(\omega^2 - K/m)^2 + \gamma^2\omega^2} = \frac{m\pi C}{2\pi\gamma K}$$

By comparing this result with (b) we can find C

$$\langle \tilde{x}^2(t) \rangle = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 = \frac{k_B T}{K}$$

$$\langle \tilde{x}^2(t) \rangle = \frac{m\pi C}{2\pi\gamma K} = \frac{k_B T}{K} \;\;\rightarrow\;\; C = \frac{2k_B T\gamma}{m}$$

and finelly

$$\nu_x(\omega) = \frac{2k_B T\gamma m}{(\omega^2 - k/m)^2 + \omega^2\gamma^2}$$

$$\nu_v(\omega) = \frac{2k_B T\gamma m\omega^2}{(\omega^2 - k/m)^2 + \omega^2\gamma^2}$$

(f) Because of the external force that couples to x the system isn't in equilibrium, namely there is dissipation

$$H = \frac{p^2}{2m} + \frac{1}{2}Kx^2 - mgx + F(t)x$$

We use Hamilton equations:

1. $\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x}$
2. $\frac{\partial H}{\partial x} = Kx - mg + F = -\dot{p} = -m\ddot{x}$

from the two equations, friction γ that interact with velocity and a random force $f(t)$ like in (d) we find a Langevin's equation

$$m\ddot{x} + m\gamma\dot{x} + Kx - mg = -F + mf(t)$$

By same way as in (e) we get $(m\omega^2 - i\omega m\gamma - K)\tilde{X}(\omega) = F(\omega)$ we rearrange a little to get the susceptibility χ_x that describes the linear response of x to F

$$\frac{\langle \tilde{X}(\omega)\rangle}{F(\omega)} = \frac{1}{m\omega^2 - K - i\omega m\gamma} = \chi_x$$

$$Im\,\chi_x = \frac{m\omega\gamma}{(m\omega^2 - k)^2 + m^2\omega^2\gamma^2}$$

By compering with $\nu_x(\omega)$ we show that FDT holds.

$$\nu_x(\omega) = \frac{2k_BT}{\omega} Im\,\chi_x = \frac{2k_BTm\gamma}{(m\omega^2 - k)^2 + m^2\omega^2\gamma^2}$$

(g) Because of the external vector potential A(t) that couples to v the system isn't in equilibrium, namely there is dissipation

$$H = \frac{p^2}{2m} + \frac{1}{2}Kx^2 - mgx + \frac{p}{m}A(t)$$

We use Hamilton equations:

1. $\frac{\partial H}{\partial p} = \frac{p}{m} + \frac{A}{m} = \dot{x}$
2. $\frac{\partial H}{\partial x} = Kx - mg = -\dot{p} = -m\ddot{x} + \dot{A}$

from the two equations, friction γ that interact with velocity and a random force $f(t)$ like in (d) we find a Langevin's equation

$$m\ddot{x} + m\gamma\dot{x} + Kx - mg = \dot{A} + mf(t)$$

By same way as in (e) we get $(-m\omega^2 - i\omega m\gamma + K)\tilde{X}(\omega) = i\omega A(\omega)$ we rearrange a little to get the susceptibility χ_v that describes the linear response of v to A

$$\frac{\langle \tilde{V}(\omega)\rangle}{A(\omega)} = i\omega\frac{\langle \tilde{X}(\omega)\rangle}{A(\omega)} = \frac{\omega^2}{m\omega^2 - K - i\omega m\gamma} = \chi_v$$

$$Im\,\chi_v = \frac{m\omega^3\gamma}{(m\omega^2 - k)^2 + m^2\omega^2\gamma^2}$$

By compering with $\nu_v(\omega)$ we show that FDT holds.

$$\nu_x(\omega) = \frac{2k_BT}{\omega} Im\,\chi_x = \frac{2k_BTm\gamma\omega^2}{(m\omega^2 - k)^2 + m^2\omega^2\gamma^2}$$