Ex8034: Brownian particle on a ring

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The problem:

The motion of a classical Brownian particle on a 1D ring is described by the Langevin equation $m\ddot{\theta} + \eta\dot{\theta} = f(t)$, where $f(t)$ is due to a noisy electromotive force that has a correlation function $\langle f(t')f(t'')\rangle = C_f(t'-t'')$. The power spectrum $\tilde{C}_f(\omega)$ is defined as the Fourier transform of the correlation function. We consider two cases:

(a) High temperature white noise $\tilde{C}_f(\omega) = \nu$.

(b) Zero temperature noise $\tilde{C}_f(\omega) = c|\omega|$.

We define the angular velocity of the particle as $v = \dot{\theta}$, and its Cartesian coordinate as $x = \sin(\theta)$. In the absence of noise the dynamics is characterized by the damping time $t_c = m/\eta$.

In items (3)-(5) you should assume a spreading scenario: the particle is initially ($t = 0$) located at $\theta \sim 0$. The spreading during the transient period $0 < t < t_c$ is assumed to be negligible. In item (6) assume that the particle had been launched in the far past ($t = -\infty$): accordingly there is no preferred location on the ring.

1. Find the exact correlation function $\langle v(t)v(0)\rangle$ in case (a).
2. Find the correlation function $\langle v(t)v(0)\rangle$ for $t \gg t_c$ in case (b).
3. Find the spreading $S(t) \equiv \langle\theta(t)^2\rangle$ for $t \gg t_c$ in case (a).
4. Find the spreading $S(t) \equiv \langle\theta(t)^2\rangle$ for $t \gg t_c$ in case (b).
5. Express $\langle x(t)^2\rangle$ for a spreading scenario given $S(t)$.
6. Express the correlation function $\langle x(t)x(0)\rangle$ given $S(t)$.
7. Write the explicit long time expression for $\langle x(t)x(0)\rangle$ in case (b), and deduce what is the critical value η_c above which a "phase transition" is expected in the response characteristics of the system.

Tips: For a Gaussian variable that has zero average $\langle e^{i\varphi}\rangle = e^{-(1/2)\langle\varphi^2\rangle}$.
The Fourier transform of $|\omega|$ has zero area, with negative tails $-1/(\pi t^2)$.
If you fail to solve (6), assume that the answer is the same as in (5), and proceed to (7).

The solution:

(1) We will start with writing the Langevin equation for the velocity $m\dot{v} + \eta v = f(t)$, we can solve it with Fourier transform:

$$(-im\omega + \eta)v_\omega = f(\omega) \quad (1)$$

$$v_\omega = \frac{f(\omega)}{\eta - im\omega} \quad (2)$$

Now we can take square absolute value from both sides and average :

$$\langle|v_\omega|^2\rangle = \frac{\langle|f(\omega)|^2\rangle}{\eta^2 + m^2\omega^2} \quad (3)$$

From the Wiener-Khinchin theorem we get that $\langle |f(\omega)|^2 \rangle = \tilde{C}_f(\omega) \times t$, so we get:

$$\tilde{C}_v(\omega) = \frac{\tilde{C}_f(\omega)}{\eta^2 + m^2\omega^2} \tag{4}$$

For case (a) $\tilde{C}_f(\omega) = \nu$ we get:

$$\tilde{C}_v(\omega) = \frac{\nu}{\eta^2 + m^2\omega^2} \tag{5}$$

After inverse Fourier transform:

$$C_v(t) = \int \frac{d\omega'}{2\pi} \frac{\nu}{\eta^2 + m^2\omega'^2} e^{-i\omega' t} = \frac{\nu}{2m\eta} \int d\omega \frac{1}{\pi} \frac{\frac{\eta}{m}}{\left(\frac{\eta}{m}\right)^2 + \omega'^2} e^{-i\omega' t} \tag{6}$$

This is a Lorentzian, so we get:

$$C_v(t) = \langle v(t)v(0) \rangle = \frac{\nu}{2m\eta} e^{-\frac{|t|}{\left(\frac{m}{\eta}\right)}} = \frac{\nu}{2m\eta} e^{-\frac{|t|}{t_c}} \tag{7}$$

(2) Now we use the same equation but in case (b) $\tilde{C}_f(\omega) = c|\omega|$:

$$\tilde{C}_v(\omega) = \frac{\tilde{C}_f(\omega)}{\eta^2 + m^2\omega^2} = \frac{c|\omega|}{\eta^2 + m^2\omega^2} \tag{8}$$

we need to do inverse Fourier transform:

$$C_v(t) = \int \frac{d\omega'}{2\pi} \frac{c|\omega'|}{\eta^2 + m^2\omega'^2} e^{-i\omega' t} \tag{9}$$

We know that the change of $|\omega|$ is slow except near the $\omega = 0$, so the shape of $\omega \approx 0$ is determined by the higher t and the shape of $\omega \gg 0$ is determined by the lower t. We take the limit $t \gg t_c$ so we can neglect $\omega > \frac{1}{t_c}$ and get:

$$C_v(t) = \langle v(t)v(0) \rangle = \int \frac{d\omega'}{2\pi} \frac{c|\omega'|}{\eta^2 + m^2\omega'^2} e^{-i\omega' t} = \frac{c}{\eta^2} \int \frac{d\omega'}{2\pi} \frac{|\omega'|}{1 + t_c^2\omega'^2} e^{-i\omega' t} \approx \frac{c}{\eta^2} \int \frac{d\omega'}{2\pi} |\omega'| e^{-i\omega' t} \tag{10}$$

We know that the Fourier transform of $|\omega|$ has zero area, with negative tails $-\frac{1}{\pi t^2}$, so we get:

$$C_v(t) = \langle v(t)v(0) \rangle \approx \frac{c}{\eta^2} \int \frac{d\omega'}{2\pi} |\omega'| e^{-i\omega' t} = -\frac{c}{\eta^2 \pi t^2} \tag{11}$$

(3) In the beginning we define $\dot{\theta} = v$, so we get:

$$\theta(t) = \int_0^t dt' v(t') \tag{12}$$

$$\theta^2(t) = \int_0^t \int_0^t dt' dt'' v(t') v(t'') \tag{13}$$

$$\langle\theta^2(t)\rangle = \int_0^t\int_0^t dt'dt''\langle v(t')v(t'')\rangle = \int_0^t\int_0^t dt'dt''C_v(t'-t'') \tag{14}$$

We can see that t', t'' are independent variables, so we can choose that $t' > t''$ and double the result. We can do a change of variables to two dependent variables $T = t' \to 0 < T < t, \tau = t' - t'' \to 0 < \tau < T$.

$$\langle\theta^2(t)\rangle = 2\int_0^t dT\int_0^T d\tau C_v(\tau) = 2\int_0^t dT\int_0^T d\tau\frac{\nu}{2m\eta}e^{-\frac{|\tau|}{\left(\frac{m}{\eta}\right)}} \tag{15}$$

The correlation decay very fast so in the limit $t \gg t_c$ we can take the integral to infinite:

$$\langle\theta^2(t)\rangle = 2\int_0^t dT\int_0^\infty d\tau\frac{\nu}{2m\eta}e^{-\frac{|\tau|}{\left(\frac{m}{\eta}\right)}} = 2t\frac{\nu}{2m\eta}\left(\frac{m}{\eta}\right) = \frac{\nu}{\eta^2}t \tag{16}$$

(4) In the same way:

$$\langle\theta^2(t)\rangle = 2\int_0^t dT\int_0^T d\tau C_v(\tau) \tag{17}$$

We can neglect the spreading during the transient period $0 < t < t_c$, so we take the limit:

$$\langle\theta^2(t)\rangle = 2\int_{t_c}^t dT\int_0^T d\tau C_v(\tau) \tag{18}$$

The solution we found to $C_v(t)$ in case (b) is good just for $t \gg t_c$, so we need to divide the integral to two parts (we assume that the limit $T = t_c$ is the lower limit to our solution):

$$\langle\theta^2(t)\rangle = 2\int_{t_c}^t dT\left(\int_0^\infty d\tau C_v(\tau) - \int_T^\infty d\tau C_v(\tau)\right) \tag{19}$$

The part $\int_0^\infty d\tau C_v(\tau) = \tilde{C}_v(\omega = 0) = 0$, so we get:

$$\langle\theta^2(t)\rangle = -2\int_{t_c}^t dT\int_T^\infty d\tau C_v(\tau) = 2\int_{t_c}^t dT\int_T^\infty d\tau\frac{c}{\eta^2\pi\tau^2} = 2\int_{t_c}^t dT\frac{c}{\eta^2\pi T} = \frac{2c}{\pi\eta^2}\ln\frac{|t|}{t_c} \tag{20}$$

(5) We defined $x = \sin\theta$:

$$\langle x^2(t)\rangle = \langle\sin^2\theta(t)\rangle = \langle\frac{\left(e^{i\theta} - e^{-i\theta}\right)^2}{-4}\rangle = \frac{1}{4}\langle\left(2 - e^{i2\theta} - e^{-i2\theta}\right)\rangle = \frac{1}{2} - \frac{1}{4}\langle e^{i2\theta}\rangle - \frac{1}{4}\langle e^{-i2\theta}\rangle \tag{21}$$

We get a tip that for a Gaussian variable that has zero average $\langle e^{i\varphi}\rangle = e^{-(1/2)\langle\varphi^2\rangle}$. Because θ is a Gaussian variable and it has zero average we get:

$$\langle x^2(t)\rangle = \frac{1}{2} - \frac{1}{4}\langle e^{i2\theta}\rangle - \frac{1}{4}\langle e^{-i2\theta}\rangle = \frac{1}{2} - \frac{1}{4}e^{-2\langle\theta^2\rangle -} - \frac{1}{4}e^{-2\langle\theta^2\rangle} = \frac{1}{2}\left(1 - e^{-2\langle\theta^2\rangle}\right) = \frac{1}{2}\left(1 - e^{-2S(t)}\right) \tag{22}$$

Note that $\langle(2\theta)^2\rangle = \langle(-2\theta)^2\rangle = 4\langle\theta^2\rangle$

(6) In the previous sections we assumed that $\theta(0) \approx 0$ and we talked about short times, so we could treat θ like a coordinate and calculate $\langle\theta(t)^2\rangle$. Now there isn't a preferred location on the ring so we can't calculate $S(t) = \langle\theta(t)^2\rangle$, because θ is not well defined. So we can't calculate $\langle x^2(t)\rangle$ like

before, just the correlation between two different times $\langle x(t)x(0)\rangle$.
By definition:

$$\langle x(t)x(0)\rangle = \int_0^{2\pi}\int_0^{2\pi} \sin(\theta_t)\sin(\theta_0)\rho(\theta_t,\theta_0)d\theta_t d\theta_0 \tag{23}$$

The formula for conditional probability is:

$$\rho(A|B) = \frac{\rho(A,B)}{\rho(B)} \to \rho(A,B) = \rho(A|B)\rho(B) \tag{24}$$

$$\langle x(t)x(0)\rangle = \int_0^{2\pi}\int_0^{2\pi} \sin(\theta_t)\sin(\theta_0)\rho(\theta_t|\theta_0)\rho(\theta_0)d\theta_t d\theta_0 \tag{25}$$

We get that in $t=0$, θ_0 has a Uniform distribution, namely $\rho(\theta_0) = \frac{1}{2\pi}$.
Additionally $\rho(\theta_t|\theta_0)$ is the probability to find θ_t when we know where is θ_0, and this is like the previous section, when we assumed that $\theta_0 = 0$.
The probability $\rho(\theta_t|\theta_0)$ depends only on the difference between θ_t and θ_0, it doesn't depends on one of them, so let's define $\delta\theta = \theta_t - \theta_0$, when $\rho(\theta_t|\theta_0)d\theta_t = \rho(\delta\theta)d\delta\theta$
By using a trigonometric identities we get:

$$\int_0^{2\pi}\int_0^{2\pi} \sin(\theta_t)\sin(\theta_0)\rho(\theta_t|\theta_0)\rho(\theta_0)d\theta_t d\theta_0 = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \left(\cos(\delta\theta) - \cos(2\theta_0+\delta\theta)\right)\rho(\delta\theta)d\delta\theta d\theta_0 = \tag{26}$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\cos(\delta\theta)\rho(\delta\theta)d\delta\theta d\theta_0 - \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\cos(2\theta_0+\delta\theta)\rho(\delta\theta)d\delta\theta d\theta_0 =$$

The first integral doesn't depend on θ_0, the integral on θ_0 in the second term give 0, and we get:

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\cos(\delta\theta)\rho(\delta\theta)d\delta\theta d\theta_0 = \frac{1}{2}\int_0^{2\pi}\cos(\delta\theta)\rho(\delta\theta)d\delta\theta = \frac{1}{2}\langle\cos(\delta\theta)\rangle \tag{27}$$

When we take $\theta_0 = 0$ we get that $\delta\theta$ is the same θ we define in the Previous section and get:

$$\frac{1}{2}\langle\cos(\delta\theta)\rangle = \frac{1}{4}\langle e^{i\delta\theta} + e^{-i\delta\theta}\rangle = \frac{1}{4}\left(\langle e^{i\delta\theta}\rangle + \langle e^{-i\delta\theta}\rangle\right) = \frac{1}{2}e^{-\frac{1}{2}\langle\delta\theta^2\rangle} = \frac{1}{2}e^{-\frac{1}{2}S(t)} \tag{28}$$

(7) For case (b):

$$S(t) = \frac{2c}{\pi\eta^2}\ln\frac{t}{t_c} \tag{29}$$

$$\langle x(t)x(0)\rangle = \frac{1}{2}e^{-\frac{1}{2}S(t)} = \frac{1}{2}\left(\frac{t}{t_c}\right)^{-\frac{c}{\pi\eta^2}} \tag{30}$$

From the FDT we get the relationship between the correlation function and the response:

$$\mathrm{Im}\chi \sim \frac{\omega}{2T}\tilde{C}_{xx}(\omega) \tag{31}$$

In the DC limit, we get:

$$\mathrm{Im}\chi = \frac{\omega}{2T}\tilde{C}_{xx}(\omega = 0) \tag{32}$$

When:

$$\tilde{C}_{xx}(\omega = 0) = \int_{-\infty}^{\infty} C_{xx}(t)dt = \int_{-\infty}^{\infty} \frac{1}{2}\left(\frac{t}{t_c}\right)^{-\frac{c}{\pi\eta^2}} dt \tag{33}$$

We define "phase transition" as When the response diverges. this will happen when $\frac{c}{\pi\eta^2} \leq 1$, so we get:

$$\eta_c = \sqrt{\frac{c}{\pi}} \tag{34}$$