E8034: Brownian particle on a ring

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The problem:

The motion of a classical Brownian particle on a 1D ring is described by the Langevin equation $m\ddot{\theta} + \eta\dot{\theta} = f(t)$, where $f(t)$ is due to a noisy electromotive force that has a correlation function $\langle f(t')f(t'')\rangle = C_f(t' - t'')$. The power spectrum $\tilde{C}_f(\omega)$ is defined as the Fourier transform of the correlation function. We consider two cases:

1. (a) High temperature white noise $\tilde{C}_f(\omega) = \nu$.
2. (b) Zero temperature noise $\tilde{C}_f(\omega) = c|\omega|$.

We define the angular velocity of the particle as $v = \dot{\theta}$, and its Cartesian coordinate as $x = \sin(\theta)$. In the absence of noise the dynamics is characterized by the damping time $t_c = m/\eta$.
In items (3)-(5) you should assume a spreading scenario: the particle is initially ($t = 0$) located at $\theta \sim 0$. The spreading during the transient period $0 < t < t_c$ is assumed to be negligible. In item (6) assume that the particle had been launched in the far past ($t = -\infty$): accordingly there is no preferred location on the ring.

1. Find the exact correlation function $\langle v(t)v(0)\rangle$ in case (a).
2. Find the correlation function $\langle v(t)v(0)\rangle$ for $t \gg t_c$ in case (b).
3. Find the spreading $S(t) \equiv \langle\theta(t)^2\rangle$ for $t \gg t_c$ in case (a).
4. Find the spreading $S(t) \equiv \langle\theta(t)^2\rangle$ for $t \gg t_c$ in case (b).
5. Express $\langle x(t)^2\rangle$ for a spreading scenario given $S(t)$.
6. Express the correlation function $\langle x(t)x(0)\rangle$ given $S(t)$.
7. Write the explicit long time expression for $\langle x(t)x(0)\rangle$ in case (b), and deduce what is the critical value η_c above which a "phase transition" is expected in the response characteristics of the system.

Tips: For a Gaussian variable that has zero average $\langle e^{i\varphi}\rangle = e^{-(1/2)\langle\varphi^2\rangle}$.
The Fourier transform of $|\omega|$ has zero area, with negative tails $-1/(\pi t^2)$.
If you fail to solve (6), assume that the answer is the same as in (5), and proceed to (7).

The solution:

1. Rewriting the equation as $m\frac{d}{dt'}[v(t')] + \eta v(t') = f(t')$, and multiplying by the equation $m\frac{d}{dt''}[v(t'')] + \eta v(t'') = f(t'')$, we get the expression:

$$m^2\frac{d}{dt'}\frac{d}{dt''}[v(t')v(t'')] + m\eta\frac{d}{dt'}[v(t')v(t'')] + m\eta\frac{d}{dt''}[v(t')v(t'')] + \eta^2[v(t')v(t'')] = f(t')f(t'')$$

now we shall take the expectation values of both sides of the equation. $f(t)$ is a stationary signal, i.e. $< f(t')f(t'') >= C_{ff}(t' - t'')$, we shall assume that $v(t)$ will also be a stationary signal: $< v(t')v(t'') >= C_{vv}(t' - t'')$. This is true when the initial velocity is "forgotten" and can be assumed to be a random variable. Now we can write the equation with a single variable $t = t' - t''$, and replace $\frac{d}{dt'}$ with $\frac{d}{dt}$ and $\frac{d}{dt''}$ with $-\frac{d}{dt}$. We get the much simpler equation:

$$-m^2 \frac{d^2}{dt^2} C_{vv}(t) + \eta^2 C_{vv}(t) = C_{ff}(t)$$

in case a, $C_{ff}(t)$ is a delta function. The simplest way to solve the equation is using a Fourier transform:

$$m^2\omega^2 \tilde{C}_{vv}(\omega) + \eta^2 \tilde{C}_{vv}(\omega) = \tilde{C}_{ff}(\omega) = \nu$$

$$\tilde{C}_{vv}(\omega) = \frac{\nu}{(m\omega)^2 + \eta^2} \Rightarrow C_{vv}(t) = \int_{-\infty}^{\infty} \frac{\nu}{(m\omega)^2+\eta^2} e^{i\omega t} \frac{d\omega}{2\pi} = \int_{-\infty}^{\infty} \frac{\nu}{(m\omega)^2+\eta^2} e^{i\omega t} \frac{d\omega}{2\pi} =$$

$$= -\oint_{r\to\infty} \frac{\nu}{2i\eta m} \frac{e^{i\omega t}}{\omega + \frac{i\eta}{m}} \frac{d\omega}{2\pi} + \oint_{r\to\infty} \frac{\nu}{2i\eta m} \frac{e^{i\omega t} - e^{-\frac{\eta t}{m}}}{\omega - \frac{i\eta}{m}} \frac{d\omega}{2\pi} + \frac{\nu}{2\eta m} e^{-\frac{\eta t}{m}} \oint_{r\to\infty} \frac{1}{\omega - \frac{i\eta}{m}} \frac{d\omega}{2\pi i} =$$

$= \frac{\nu}{2\eta m} e^{-\frac{\eta t}{m}}$

Where the orbital integral is over a semi-circle with infinite radius in the upper semi-plain $Im[\omega] \geq 0$, and the last equality is due to the Residue Theorem.

2. In case b, the previous method would fail, since $\tilde{C}_{ff}(\omega) = C|\omega|$ is non-analytical at $\omega = 0$, the Residue Theorem will not work, and the inverse Fourier transform is hard to calculate. However, since we are only asked for the solution at very long times, much longer than the system's decay time $t_c = m/\eta$, we can assume that the solution is now slowly-varying, i.e. $\frac{d^2}{dt^2} C_{vv}(t) \ll \frac{1}{t_c^2} C_{vv}(t)$. Therefore, the solution for long times, is

$$C_{vv}(t) \simeq \frac{1}{\eta^2} C_{ff}(t) = -\frac{C}{\eta^2 \pi t^2}$$

3. Again, we are only asked for the solution at long times, $t \gg t_c$. Therefore we can go back to the equation $C_{vv}(t) \simeq \frac{1}{\eta^2} C_{ff}(t)$. Now we can write: $S(t) = <(\theta(t))^2> = <\int_0^t v(t')dt' \int_0^t v(t'')dt''> = \int_0^t d\tilde{t} \int_{-\tilde{t}}^{\tilde{t}} \frac{1}{\eta^2} C_{ff}(\tau) d\tau$. In the last equality the variables $\tilde{t} = t' + t''$ and $\tau = t' - t''$ were used. In case a, $S(t) = \int_0^t d\tilde{t} \int_{-\tilde{t}}^{\tilde{t}} \frac{1}{\eta^2} \nu \delta(\tau) d\tau = \frac{\nu}{\eta^2} t$

4. $S(t) = \int_0^t d\tilde{t} \int_{-\tilde{t}}^{\tilde{t}} \frac{1}{\eta^2} C_{ff}(\tau) d\tau$. We know how the function $C_{ff}(\tau)$ looks like for a large τ. However, since the function is symmetric and has zero area ($\tilde{C}_{ff}(\omega = 0) = 0$) , $\int_{-\tilde{t}}^{\tilde{t}} \frac{1}{\eta^2} C_{ff}(\tau) d\tau \equiv f(\tilde{t}) = \frac{2}{\eta^2} \int_0^{\tilde{t}} C_{ff}(\tau) d\tau = \frac{2}{\eta^2} \int_0^{\infty} C_{ff}(\tau) d\tau - \frac{2}{\eta^2} \int_{\tilde{t}}^{\infty} C_{ff}(\tau) d\tau = -\frac{2}{\eta^2} \int_{\tilde{t}}^{\infty} (-\frac{C}{\pi \tau^2}) d\tau = \frac{2C}{\eta^2 \pi \tilde{t}}$. Now we need to calculate $\int_0^t f(\tilde{t}) d\tilde{t}$ for large t. Since we only know the form of $f(\tilde{t})$ for large $\tilde{t}$, we can write $S(t) = S(t_c) + \int_{t_c}^t \frac{2C}{\eta^2 \pi \tilde{t}} d\tilde{t} = S(t_c) + \frac{2C}{\eta^2 \pi} \ln(\frac{t}{t_c})$, where $S(t_c)$ is some constant. However, if we assume, as instructed, that the spreading for $0 < t < t_c$ is neglible, then $S(t_c) = 0$, and $S(t) = \frac{2C}{\eta^2 \pi} \ln(\frac{t}{t_c})$

5. For a brownian motion, the noise $f(t)$ is a gaussian random variable, and so $\theta(t)$ is also a gaussian random variable. Therefore:

$$<(x(t))^2> = <sin^2(\theta(t))> = <\left(\frac{e^{i\theta(t)} - e^{-i\theta(t)}}{2i}\right)^2> = \frac{1}{4}\left[2 - <e^{2i\theta(t)}> - <e^{-2i\theta(t)}>\right] =$$

$$= \frac{1}{4}\left[2 - < e^{2i\theta(t)} > - < e^{-2i\theta(t)} >\right] = \frac{1}{4}\left[2 - e^{-2\langle(\theta(t))^2\rangle} - e^{-2\langle(\theta(t))^2\rangle}\right] = \frac{1}{2}\left[1 - e^{-2S(t)}\right]$$

6. Now we are no longer dealing with a spreading scenario. We have to calculate $< x(t)x(0) >$where $x(0)$ is also a random variable.

$$< x(t)x(0) >=< sin(\theta(t))sin(\theta(0)) >=< \left(\frac{e^{i\theta(t)} - e^{-i\theta(t)}}{2i}\right)\left(\frac{e^{i\theta(0)} - e^{-i\theta(0)}}{2i}\right) >=$$

$$= -\frac{1}{4} < \left(e^{i(\theta(t)+\theta(0))} + e^{i(\theta(t)+\theta(0))} - e^{i(\theta(t)-\theta(0))} - e^{i(\theta(t)-\theta(0))}\right) >=$$

$$= \frac{1}{2}\left[e^{-\frac{1}{2}<(\theta(t)-\theta(0))^2>} - e^{-\frac{1}{2}<(\theta(t)+\theta(0))^2>}\right]$$

Using the law of total expectation:

$$< (\theta(t) - \theta(0))^2 >=<< (\theta(t) - \theta(0))^2|\theta(0) >>=< S(t) >= S(t)$$

$$< (\theta(t) + \theta(0))^2 >=<< (\theta(t) - \theta(0))^2|\theta(0) > +4 < \theta(t)\theta(0)|\theta(0) >= S(t) + 4 < \theta^2(0) >$$

Since the particle was lounched at $t \longrightarrow -\infty$, we can write the equality as:

$$< (\theta(t)+\theta(0))^2 >= S(t)+4 < \theta^2(0)-\theta^2(-\infty) > +4\theta^2(-\infty) = S(t)+4S(\infty)+Const \to \infty$$

where the expression $S(\infty) = \lim_{t\to\infty}(s(t))$ diverges. Therefore:

$$< x(t)x(0) >= \frac{1}{2}\left[e^{-\frac{1}{2}<(\theta(t)-\theta(0))^2>} - e^{-\frac{1}{2}<(\theta(t)+\theta(0))^2>}\right] = \frac{1}{2}e^{-\frac{1}{2}S(t)}$$

7. $< x(t)x(0) >= \frac{1}{2}e^{-\frac{1}{2}S(t)} = \frac{1}{2}e^{-\frac{1}{2}\frac{2C}{\eta^2\pi}\ln(\frac{t}{t_c})} = \frac{1}{2}(\frac{t_c}{t})^{\frac{C}{\eta^2\pi}}$. For $\frac{C}{\eta^2\pi} < 1 \to \eta > \eta_c = \sqrt{\frac{C}{\pi}}$, the correlation between $X(t)$ and $X(0)$ is strong, which means that the particle can be thought of as "localized". In the oposite case, when $\frac{C}{\eta^2\pi} > 1 \to \eta < \sqrt{\frac{C}{\pi}}$, the correlation is weak. One response charactaristic that experiences a "phase transition" is the integral $\int_{t_c}^{\infty} < x(t)x(0) > dt$ which converges only for $\eta < \eta_c$.