Exercises in Statistical Mechanics

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This exercises pool is intended for a graduate course in "statistical mechanics". Some of the problems are original, while other were assembled from various undocumented sources. In particular some problems originate from exams that were written by B. Horovitz (BGU), S. Fishman (Technion), and D. Cohen (BGU).

====== [Exercise 8034]

Brownian particle on a ring

The motion of a classical Brownian particle on a 1D ring is described by the Langevin equation $\mathsf{m}\ddot{\theta} + \eta\dot{\theta} = f(t)$, where $f(t)$ is due to a noisy electromotive force that has a correlation function $\langle f(t')f(t'')\rangle = C_f(t'-t'')$. The power spectrum $\tilde{C}_f(\omega)$ is defined as the Fourier transform of the correlation function. We consider two cases:

(a) High temperature white noise $\tilde{C}_f(\omega) = \nu$.

(b) Zero temperature noise $\tilde{C}_f(\omega) = c|\omega|$.

We define the angular velocity of the particle as $v = \dot{\theta}$, and its Cartesian coordinate as $x = \sin(\theta)$. In the absence of noise the dynamics is characterized by the damping time $t_c = \mathsf{m}/\eta$.

In items (3)-(5) you should assume a spreading scenario: the particle is initially ($t = 0$) located at $\theta \sim 0$. The spreading during the transient period $0 < t < t_c$ is assumed to be negligible. In item (6) assume that the particle had been launched in the far past ($t = -\infty$): accordingly there is no preferred location on the ring.

1. Find the exact correlation function $\langle v(t)v(0)\rangle$ in case (a).
2. Find the correlation function $\langle v(t)v(0)\rangle$ for $t \gg t_c$ in case (b).
3. Find the spreading $S(t) \equiv \langle \theta(t)^2\rangle$ for $t \gg t_c$ in case (a).
4. Find the spreading $S(t) \equiv \langle \theta(t)^2\rangle$ for $t \gg t_c$ in case (b).
5. Express $\langle x(t)^2\rangle$ for a spreading scenario given $S(t)$.
6. Express the correlation function $\langle x(t)x(0)\rangle$ given $S(t)$.
7. Write the explicit long time expression for $\langle x(t)x(0)\rangle$ in case (b), and deduce what is the critical value η_c above which a "phase transition" is expected in the response characteristics of the system.

Tips: For a Gaussian variable that has zero average $\langle e^{i\varphi}\rangle = \exp[-(1/2)\langle\varphi^2\rangle]$.
The Fourier transform of $|\omega|$ has zero area, with negative tails $-1/(\pi t^2)$.
If you fail to solve (6), assume that the answer is the same as in (5), and proceed to (7).