Ex8032: Sub diffusion of Brownian particle

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The problem:

The motion of a brownian particle in 1D is given by the Hamiltonian:

$$H_{Total}(x,p,A(t)) = \frac{1}{2m}(p - A(t))^2 + H_{Bath}(x)$$

and the equation of motion is:

$$m\frac{\partial < v >}{\partial t} = -\eta < v > + f(t)$$

1. Define $f(t)$ from $A(t)$.
2. What is the generalised susceptability χ. (of v driven by A)
3. Find the power spectrum $\tilde{C}(\omega)$ of the velocity v.
4. Find the explicit expression for the correlation function $C(\tau)$ in the limit of high and low temperatures.

   For the following questions assume $T = 0$.
5. Find the coefficient C_0 in $C(\tau) \sim -C_0/\tau^2$.
6. Given the definition $S(t) = \left\langle (x(t) - x(0))^2 \right\rangle$, Find an expression for $\frac{dS(t)}{dt}$, using the correlation function $C(\tau)$.
7. Given $S(t_0) = S_0$, find $S(t)$ for $t > t_0$.

Solution

1. Hamilton equations give:

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{1}{m}(p - A)$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial H_{bath}}{\partial x}$$

They give a straightforward solution (equation of motion):

$$m\frac{\partial \dot{x}}{\partial t} = -\frac{\partial H_{bath}}{\partial x} - \frac{\partial A}{\partial t}$$

The two terms on the right can then be easily identified:

$$f(t) = -\frac{\partial A}{\partial t}$$

$$\eta \dot{x} = \frac{\partial H_{bath}}{\partial x}$$

2. Fourier transforming the equation of motion results in:

$$(\eta - i\omega m)\, v_\omega = i\omega A_\omega$$

and if we define $\tau_\eta = m/\eta$ we get the we get:

$$\chi(\omega) = \frac{1}{\eta}\frac{i\omega}{1 - i\omega\tau_\eta}$$

3. The FD relations are:

$$\frac{\mathrm{Im}\,[\chi]}{\omega} = \frac{1}{\hbar\omega}\tanh\left(\frac{\hbar\omega}{2T}\right)\cdot\tilde{C}(\omega)$$

and from (2) it is known that $\frac{\mathrm{Im}[\chi]}{\omega} = \frac{1}{\eta}\frac{1}{1+(\omega\tau_\eta)^2}$ we therefore get the power spectrum of v to be:

$$\tilde{C}(\omega) = \frac{1}{\eta}\frac{\hbar\omega}{1 + (\omega\tau_\eta)^2}\coth\left(\frac{\hbar\omega}{2T}\right)$$

4. In order to obtain the high and low temperature limits we note the behaviour of $\coth(x)$ at the requested limits:

$$\coth(x) \underset{x\to 0}{\approx} \frac{1}{x}$$

$$\coth(x) \underset{|x|\to\infty}{\approx} sign\,(x)$$

we then can get for high temperatures ($T \gg \frac{1}{2}\hbar\omega$):

$$\tilde{C}(\omega) = \frac{1}{\eta}\frac{2T}{1 + (\omega\tau_\eta)^2}$$

and for the low temperatures ($T \ll \frac{1}{2}\hbar\omega$):

$$\tilde{C}(\omega) = \frac{1}{\eta}\frac{\hbar\,|\omega|}{1 + (\omega\tau_\eta)^2}$$

The correlation function is the inverse Fourier Transform of $\tilde{C}(\omega)$.
a) For high temperatures we get:

$$C(\tau) = \int\limits_{-\infty}^{+\infty}\frac{2T}{\eta}\frac{d\omega}{2\pi}\frac{\exp[i\omega\tau]}{1 + (\omega\tau_\eta)^2}$$

and setteing $\omega\tau_\eta = x$ we get the integral:

$$C(\tau) = \frac{T}{\pi m}\int\limits_{-\infty}^{\infty}\frac{\exp[ix\tau/\tau_\eta]\,dx}{1 + x^2}$$

using the residue theorm the integral is easily solvable:

$$C(\tau) = \frac{T}{m}\exp\left[-\frac{\tau}{\tau_\eta}\right]$$

b) For low temperature the FT gives:

$$FT(\tilde{C}(\omega)) = \int \frac{\exp\left[i\omega\tau\right]}{2\pi\eta} \frac{\hbar\left|\omega\right| d\omega}{1+\left(\omega\tau_\eta\right)^2}$$

We recognize the expression above is a multiplication of $|w|$ and the Lorentzian.
The Fourier transfrom of $|w|$ is:

$$FT(|\omega|) = -\frac{1}{\pi\tau^2}$$

And, the Fourier transfron of the Lorentzian is:

$$FT(\frac{1}{1+\left(\omega\tau_\eta\right)^2}) = e^{-\gamma\tau}$$

Now we would like to make use of the convolution theorem, so:

$$FT(\frac{|w|}{1+\left(\omega\tau_\eta\right)^2}) = -\frac{1}{\pi\tau^2} * e^{-\gamma\tau}$$

Taking into account the contribution of the tales of $e^{-\gamma\tau}$ are non significant compared to $-\frac{1}{\pi\tau^2}$ one concludes:

$$C(\tau) = -\frac{\hbar}{\pi\eta\tau^2}$$

5. since $T = 0$ we can take the low Temp. approximation in (4) and we have:

$$C_0 = \frac{\hbar}{\pi\eta}$$

6. The average displacement $x(t) - x(0)$ is zero. However the second moment is:

$$S(t) = \int_0^t \int_0^t \left\langle v(t'')v(t')\right\rangle dt'dt'' = \frac{1}{\eta^2} \int_0^t \int_0^t \left\langle f(t'')f(t')\right\rangle dt'dt'' = \frac{1}{\eta^2} \int_0^t \int_0^t C(t''-t')dt'dt''$$

The last equality is valid due to (as seen in class section 15.3) $f(t)$ being a stochastic variable and satisfying $\langle f(t)\rangle = 0$.

After a change of variables we receive:

$$S(t) = \frac{1}{\eta^2} \int_0^t \int_{-t'}^{t'} C(\tau)dt'd\tau = \int_0^t dt' \int_{-t'}^{t'} -\frac{\hbar}{\pi\eta} \cdot \frac{d\tau}{\tau^2}$$

So the derivative is:

$$\frac{d}{dt}S(t) = \int_0^t C(\tau)d\tau = \frac{\hbar}{\pi\eta}\frac{1}{t}$$

7. Using the result of (6) we get:

$$S(t) - S(t_o) = \int_{t_0}^{t} \frac{dt}{t} \frac{\hbar}{\pi\eta}$$

and finally:

$$S(t) = S(t_0) + \frac{\hbar}{\pi\eta} \ln \left[\frac{t}{t_0} \right]$$