E8032: Sub diffusion of Brownian particle

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Question

The motion of a brownian particle in 1D is given by the Hamiltonian:

$$H_{Total}(x, p; A(t)) = \frac{1}{2m}(p - A(t))^2 + H_{Bath}(x)$$

and the equation of motion is:

$$m\frac{\partial < v >}{\partial t} = -\eta < v > + f(t)$$

1. Define $f(t)$ from $A(t)$.
2. What is the generalised susceptability χ. (of v driven by A)
3. Find the power spectrum $\tilde{C}(\omega)$ of the velocity v.
4. Find the explicit expression for the correlation function $C(\tau)$ in the limit of high and low temperatures.

   For the following questions assume $T = 0$.
5. Find the coefficient C_0 in $C(\tau) \sim -C_0/\tau^2$.
6. Given the definition $S(t) = \left\langle (x(t) - x(0))^2 \right\rangle$, Find an expression for $\frac{dS(t)}{dt}$, using the correlation function $C(\tau)$.
7. Given $S(t_0) = S_0$, find $S(t)$ for $t > t_0$.

Solution

1. Hamilton equations give:

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{1}{m}(p - A)$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial H_{bath}}{\partial x}$$

They give a straightforward solution (equation of motion):

$$m\frac{\partial \dot{x}}{\partial t} = -\frac{\partial H_{bath}}{\partial x} - \frac{\partial A}{\partial t}$$

The two terms on the right can then be easily identified (taking an ensamble average):

$$f(t) = -\frac{\partial A}{\partial t}$$

$$\eta = \frac{\partial H_{bath}}{\partial x}$$

2. Fourier transforming the equation of motion results in:

$$(\eta - i\omega m)\, v_\omega = i\omega A_\omega$$

and if we define $\tau_\eta = m/\eta$ we get the we get:

$$\chi(\omega) = \frac{1}{\eta}\frac{i\omega}{1 - i\omega\tau_\eta}$$

3. The FD relations are:

$$\frac{\mathrm{Im}\,[\chi]}{\omega} = \frac{1}{\hbar\omega}\tanh\left(\frac{\hbar\omega}{2K_BT}\right)\cdot \tilde{C}(\omega)$$

and from (2) it is known that $\frac{\mathrm{Im}[\chi]}{\omega} = \frac{1}{\eta}\frac{1}{1+(\omega\tau_\eta)^2}$ we therefore get the power spectrum of v to be:

$$\tilde{C}(\omega) = \frac{1}{\eta}\frac{\hbar\omega}{1+(\omega\tau_\eta)^2}\coth\left(\frac{\hbar\omega}{2K_BT}\right)$$

4. In order to obtain the high and low temperature limits we note the behaviour of $\coth(x)$ at the requested limits:

$$\coth(x) \underset{x\to 0}{\approx} \frac{1}{x}$$

$$\coth(x) \underset{|x|\to\infty}{\approx} sign\,(x)$$

we then can get for high temperatures ($K_BT \gg \frac{1}{2}\hbar\omega$):

$$\tilde{C}(\omega) = \frac{1}{\eta}\frac{2K_BT}{1+(\omega\tau_\eta)^2}$$

and for the low temperatures ($K_BT \ll \frac{1}{2}\hbar\omega$):

$$\tilde{C}(\omega) = \frac{1}{\eta}\frac{\hbar\,|\omega|}{1+(\omega\tau_\eta)^2}$$

The correlation function is the inverse Fourier Transform of $\tilde{C}(\omega)$.
a) For high temperatures we get:

$$C(\tau) = \int_{-\infty}^{+\infty}\frac{2K_BT}{\eta}\frac{d\omega}{2\pi}\frac{\exp[i\omega\tau]}{1+(\omega\tau_\eta)^2}$$

and setteing $\omega\tau_\eta = x$ we get the integral:

$$C(\tau) = \frac{K_BT}{\pi m}\int_{-\infty}^{\infty}\frac{\exp[ix\tau/\tau_\eta]\,dx}{1+x^2}$$

using the residue theorm the integral is easily solvable:

$$C(\tau) = \frac{K_BT}{m}\exp\left[-\frac{\tau}{\tau_\eta}\right]$$

b) For low temperature the FT gives:

$$C(\tau) = -\frac{\hbar}{\pi\eta} \cdot \frac{1}{\tau^2}$$

5. since $T = 0$ we can take the low Temp. approximation in (4) and we have:

$$C_0 = \frac{\hbar}{\pi\eta}$$

6. Since the relations between the autocorrelation of v and x is (for $t = t_2 - t_1$):

$$C^{xx}(t) = \int_0^{t_2}\int_0^{t_1} \langle v(x)v(y)\rangle \, dxdy = \int_0^{t_2}\int_0^{t_1} \langle C^{vv}(x-y)\rangle \, dxdy$$

after some calculations we get:

$$\frac{d}{dt}S(t) = \int_0^t C(\tau)d\tau = \frac{\hbar}{\pi\eta}\frac{1}{t}$$

7. Using the result od (6) we get:

$$\int_{t_0}^{t} dS = \int_{t_0}^{t} \frac{dt}{t}\frac{\hbar}{\pi\eta}$$

and finally:

$$S(t) = S(t_0) + \frac{\hbar}{\pi\eta}\ln\left[\frac{t}{t_0}\right]$$