E8025: Thermal flow via a Brownian particle (Exam 2011)

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The problem:

A Brownian particle in one dimension that has mass $\mathsf{m} = 1$, is in contact with two baths: A hot bath that has temperature T_2 that induces friction with coefficient γ_2, and a cold bath that has temperature T_1 that induces friction with coefficient γ_1. Accordingly the motion of the particle is described by a Langevin equation that includes two friction terms and two independent white noise terms $f_1(t)$ and $f_2(t)$. The purpose of this question is to calculate the rate of heat flow $\dot{Q}$ from the hot to the cold bath.

Note: Each bath exerts on the particle a force that has two components: a systematic "friction" component plus a fluctuating component. The rate of heat flow $\dot{Q}$ equals the rate of work which is done by the force that is exerted on the particle by the hot bath. In steady state, on the average, it equals in absolute value to the rate of work which is done by the force that is exerted on the particle by the cold bath.

(1) Write the Langevin equation for the velocity $v(t)$. Specify the intensity of the noise terms.

(2) Find the steady state value of $\langle v^2 \rangle$.

(3) Express the instantaneous $\dot{Q}$ at time t, given $v(t)$ and $f_2(t)$.

(4) Find an expression for $\langle \dot{Q} \rangle$ at steady state.

The solution:

(1) Langevin equation for the velocity is

$$\dot{v} = \gamma_1 v + \gamma_2 v + f_1(t) + f_2(t)$$

where the intensity for white noise

$$\langle f_i(t') f_j(t'') \rangle = 2T\gamma_i \delta_i j \delta(t' - t'')$$

we define

$$\nu = \nu_1 + \nu_2 = 2\gamma_1 T_1 + 2\gamma_2 T_2$$

(2) For

$$\dot{v} = \gamma v + f(t)$$

we get

$$v(t) = \int_0^t dt' f(t') e^{-\gamma(t-t')} = \int_{-\infty}^t dt' f(t') e^{-\gamma(t-t')} + c \rightarrow c = v(-\infty) = 0$$

and now

$$\langle v^2 \rangle = \int_{-\infty}^t \int_{-\infty}^t dt' dt'' e^{-\gamma(t-t')} e^{-\gamma(t-t'')} \underbrace{\langle f(t') f(t'') \rangle}_{\nu\delta(t'-t'')} = \{t \rightarrow \infty\} = \frac{\nu}{2\gamma} = \frac{\gamma_1 T_1 + \gamma_2 T_2}{\gamma_1 + \gamma_2}$$

(3)

$$\dot{Q} = \dot{W} = Fv = \{T_2 > T_1\} = F_2 v(t) = (\gamma_2 v(t) + f_2(t))v(t)$$

(4)

$$\langle v(t) f_2(t) \rangle = \int_0^t dt' e^{-\gamma(t-t')} (\underbrace{\langle f_2(t) f_1(t') \rangle}_{0} + \underbrace{\langle f_2(t) f_2(t') \rangle}_{\nu_2 \delta(t-t')}) = \nu_2 * \frac{1}{2}$$

$$\langle \dot{Q} \rangle = \langle (\gamma_2 v(t) + f_2(t)) v(t) \rangle = \gamma_2 \langle v(t)^2 \rangle + \langle v(t) f_2(t) \rangle = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} (T_2 - T_1)$$

for

$$T_2 = T_1 \rightarrow \langle \dot{Q} \rangle = 0$$