E7050: The Drude formula

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The problem:

Consider a ring of length L, with a particle that has the Drude velocity-velocity correlation function with a time constant τ. The temperature is T.

(a) Find the conductance of the ring using the canonical FDT.

(b) What is the conductance if there are N fermions at zero temperature instead of a single particle.

(c) What is τ, and hence what is the conductance, if the scattering in the ring is due to a stochastic segment that has a transmission g.

The solution:

(a) The current obeys $I = -\frac{\partial H}{\partial \Phi}$ and so I is the "generalized force". We define

$$I = -G\dot{\Phi} \tag{1}$$

where G is a dissipation term called the "Drude conductance". Using the fluctuation-dissipation theorem we have

$$G = \frac{1}{2T}\tilde{C}(\omega = 0) = \frac{1}{2T}\langle I(t)I(0)\rangle = \frac{1}{2T}\left(\frac{e}{L}\right)^2 \langle v(t)v(0)\rangle \tag{2}$$

where in the last term we used $I = \frac{e}{L}v$. Now, the velocity-velocity correlation function is, by the Drude assumption,

$$\langle v(t)v(0)\rangle = \int_{-\infty}^{\infty} {v_T}^2 e^{-|t|/\tau} dt = 2{v_T}^2\tau \tag{3}$$

We can also express the thermal velocity from the equipartition theorem

$$\frac{1}{2}m\langle v^2\rangle = \frac{1}{2}T \quad \Longrightarrow \quad \langle v^2\rangle = \frac{T}{m} \tag{4}$$

putting this all together, we get

$$G = \frac{1}{2T}\left(\frac{e}{L}\right)^2 \cdot 2\frac{T}{m}\tau = \frac{1}{L^2}\frac{e^2}{m}\tau \tag{5}$$

Note that this is independent of T.

(b) For a system of N Fermions, the correct result is simply N times the single-particle conductance as we show below:

$$G^{[N]} = \frac{N}{L^2}\frac{e^2}{m}\tau \tag{6}$$

The only difference is in the time constant τ, which we assume here to be the same.

This is actually a result of 2 effects that exactly cancel each other out: The Fermion nature of the system and the quantum-mechanical correlation function. A quantum mechanical calculation shows that for Fermions

$$\tilde{C}^{[N]}(\omega) = \omega \coth\left(\frac{\omega}{2T}\right) \tilde{C}_{E_F}(\omega) \tag{7}$$

The conductance is given by FDT

$$\begin{aligned} G^{[N]} &= \frac{1}{\omega} \tanh\left(\frac{\omega}{2T}\right) \tilde{C}^{[N]}(\omega) \\ &= \frac{1}{\omega} \tanh\left(\frac{\omega}{2T}\right) \cdot \omega \coth\left(\frac{\omega}{2T}\right) \tilde{C}_{E_F}(\omega) \\ &= \tilde{C}_{E_F}(\omega) = \frac{N}{L^2} \frac{e^2}{m} \tau_F \end{aligned}$$

where at $T = 0$ we have $\tau_F = \tau$.

(c) We take the time constant to be $\tau = \frac{\ell}{v}$ where ℓ is the mean free path. The correlation function would then have an exponential decay $\exp[-v|t|/\ell]$ as usual. We recognize, however, that after each "scattering" event the current is a fraction $g - (1-g) = 2g - 1$ of the original (with a combination of clockwise and anticlockwise currents), and the time between adjacent scattering events is $\frac{L}{v}$, so the correlation function takes the form of a discrete sum

$$C(t) = e^2 \frac{v}{L} \sum_{n=-\infty}^{\infty} (2g-1)^n \delta\left(t - \left(\frac{L}{v} n\right)\right) \tag{8}$$

from which we get by integration

$$\frac{\ell}{L} = \frac{g}{g-1} \tag{9}$$

and the conductance in the ring is given by

$$G = \left(\frac{e^2}{mvL}\right) \frac{\ell}{L} = \left(\frac{g}{g-1}\right) \frac{e^2}{mvL} \tag{10}$$