Ex7041: FDT for RLC circuit

Submitted by: Afik Shachar

The problem:

An electrical circuit has in series components with capacitance C, inductance L, resistance R and a voltage source $V_0 \cos\omega t$ with frequency ω.

(a) Identify the responsefunction $\alpha_Q(\omega) = \langle Q(\omega)\rangle/(\frac{1}{2}V_0)$. Use this to write the energy dissipation rate.

(b) Use the fluctuation dissipation relation to identify the Fourier transform $\Phi_Q(\omega)$ of the charge correlation function. Evaluate $\langle Q^2(t)\rangle$ and compare with the result from equipartition.

(c) Evaluate the current fluctuations $\langle I^2(t)\rangle$ and compare with the result from equipartition. Under what conditions does one get Nyquist's result $\langle I^2\rangle_{\omega_1\leftrightarrow\omega_2} = \frac{2k_BT}{\pi R}(\omega_2 - \omega_1)$?

Hint: $\int_{-\infty}^{\infty} \frac{d\omega/2\pi}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2} = \frac{1}{2\gamma\omega_0^2}$, $\int_{-\infty}^{\infty} \frac{\omega^2 d\omega/2\pi}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2} = \frac{1}{2\gamma}$.

The solution:

(a)

this RLC circuit is discribed by the equation:

$$L\langle\ddot{Q}\rangle + R\langle\dot{Q}\rangle + \frac{1}{C}\langle Q\rangle = V_0 \cos(\omega t)$$

applying Fourier transform:

$$-L\Omega^2\langle Q(\Omega)\rangle - i\Omega\langle Q(\Omega)\rangle + \frac{1}{C}\langle Q(\Omega)\rangle = V_0\sqrt{\frac{\pi}{2}}\Big[\delta(\Omega-\omega) + \delta(\Omega+\omega)\Big]$$

$$\langle Q(\Omega)\rangle = \chi(\Omega)FT[V(t)] = \frac{V_0\sqrt{\frac{\pi}{2}}\big[\delta(\Omega-\omega) + \delta(\Omega+\omega)\big]}{\left(\frac{1}{C} - L\Omega^2\right) - i\Omega R}$$

$$\langle Q(\omega)\rangle = V_0\sqrt{\frac{\pi}{2}}\frac{1}{\left(\frac{1}{C} - L\omega^2\right) - i\omega R} = V_0\sqrt{\frac{\pi}{2}}\frac{\left(\frac{1}{C} - L\omega^2\right) + i\omega R}{\left(\frac{1}{C} - L\omega^2\right)^2 + \omega^2R^2}$$

$$\alpha_Q(\omega) = \frac{\langle Q(\omega)\rangle}{\frac{1}{2}V_0} = \sqrt{2\pi}\frac{\left(\frac{1}{C} - L\omega^2\right) + i\omega R}{\left(\frac{1}{C} - L\omega^2\right)^2 + \omega^2R^2}$$

$$\chi(\omega) = \frac{\alpha_Q(\omega)}{\sqrt{2\pi}} = \frac{\left(\frac{1}{C} - L\omega^2\right) + i\omega R}{\left(\frac{1}{C} - L\omega^2\right)^2 + \omega^2R^2}$$

and the energy dissipation rate (from lecture notes, sec. 12.2, eq. 619):

$$\dot{\mathcal{W}} = \frac{1}{2}\eta(\omega)V_0^2\omega^2 = \frac{1}{2}\omega^2V_0^2\frac{Im\left[\chi(\omega)\right]}{\omega} = \frac{1}{2}V_0^2\frac{\omega^2R}{\left(\frac{1}{C} - L\omega^2\right)^2 + \omega^2R^2}$$

we denote $\gamma = \frac{R}{L}$ and $\omega_0^2 = \frac{1}{CL}$

$$\dot{\mathcal{W}} = \frac{1}{2L}\frac{\gamma V_0^2\omega^2}{\left(\omega - \omega_0^2\right)^2 + \gamma^2\omega^2}$$

(b)

$$\Phi_Q(\omega) = FT\left[\langle Q(0)Q(\tau)\rangle\right]$$

from FDT:

$$\frac{Im\left[\chi(\omega)\right]}{\omega} = \frac{1}{\hbar\omega}\tanh\left(\frac{\hbar\omega}{2k_BT}\right)\Phi_Q(\omega)$$

for $\hbar\omega << k_BT$:

$$\frac{Im\left[\chi(\omega)\right]}{\omega} = \frac{1}{2k_BT}\Phi_Q(\omega)$$

$$\Phi_Q(\omega) = \frac{1}{L}\frac{2k_BT\frac{R}{L}}{\left(\frac{1}{CL}-\omega^2\right)^2+\omega^2\frac{R^2}{L^2}}$$

again $\gamma = \frac{R}{L}$ and $\omega_0^2 = \frac{1}{CL}$

$$\Phi_Q(\omega) = \frac{1}{L}\frac{2k_BT\gamma}{\left(\omega^2-\omega_0^2\right)^2+\omega^2\gamma^2}$$

and we recall that:

$$\langle Q(0)Q(\tau)\rangle = \int_{-\infty}^{\infty}\Phi_Q(\omega)\exp^{-i\omega\tau}\frac{d\omega}{2\pi}$$

therefore:

$$\langle Q^2(t)\rangle = \int_{-\infty}^{\infty}\Phi_Q(\omega)\frac{d\omega}{2\pi} = \frac{2k_BT\gamma}{L}\int_{-\infty}^{\infty}\frac{d\omega/2\pi}{\left(\omega^2-\omega_0^2\right)^2+\omega^2\gamma^2} = \frac{2k_BT\gamma}{2\gamma\omega_0^2L}$$

$$\langle Q^2(t)\rangle = Ck_BT$$

from equipartition we have

$$\left\langle\frac{1}{2}CV^2\right\rangle = \left\langle\frac{1}{2}\frac{Q^2}{C}\right\rangle = \frac{1}{2}k_BT$$

we then get the same result:

$$\left\langle Q^2(t)\right\rangle = Ck_BT$$

(c)

$$I = \dot{Q}$$

$$I(\omega) = -i\omega Q(\omega)$$

therefore

$$\Phi_I(\omega) = \omega^2\Phi_Q(\omega) = \frac{1}{L}\frac{2k_BT\gamma\omega^2}{\left(\omega^2-\omega_0^2\right)^2+\omega^2\gamma^2}$$

$$\langle I^2(t)\rangle = \int_{-\infty}^{\infty}\Phi_I(\omega)\frac{d\omega}{2\pi} = \frac{2k_BT\gamma}{L}\int_{-\infty}^{\infty}\frac{\omega^2 d\omega/2\pi}{\left(\omega^2-\omega_0^2\right)^2+\omega^2\gamma^2} = \frac{2k_BT\gamma}{2\gamma L}$$

$$\langle I^2(t)\rangle = \frac{k_B T}{L}$$

from equipartition we have

$$\left\langle \frac{1}{2} L I^2 \right\rangle = \frac{1}{2} k_B T$$

we then get the same result:

$$\left\langle I^2(t) \right\rangle = \frac{k_B T}{L}$$

to get Nyquist's result, we look again at $\Phi_I(\omega)$

$$\Phi_I(\omega) = \frac{1}{L} \frac{2k_B T\gamma}{\omega^2 \left(1 - \frac{\omega_0^2}{\omega^2}\right)^2 + \gamma^2}$$

it can be seen that for $\omega \left(1 - \frac{\omega_0^2}{\omega^2}\right) << \frac{R}{L}$, this becomes

$$\Phi_I(\omega) = \frac{2k_B T}{R}$$

which in turn gives

$$\left\langle I^2 \right\rangle_{\omega_1 \leftrightarrow \omega_2} = 2 \int_{\omega_1}^{\omega_2} \Phi_I(\omega) \frac{d\omega}{2\pi} = \frac{2k_B T}{\pi R} \int_{\omega_1}^{\omega_2} d\omega$$

$$\left\langle I^2 \right\rangle_{\omega_1 \leftrightarrow \omega_2} = \frac{2k_B T}{\pi R} \left(\omega_2 - \omega_1\right)$$