E7040: Nyquist theory for a ring RL-circuit

Submitted by: Nimrod Cohen

Question
Derive the Nyquist expression for the current-current correlation function in a closed ring, taking into account its inductance.

Use the following procedure:

1. Cite an expression for the Inductance L of the ring given its radius R and its section radius r when $R > r$.
2. Write the R-L circuit equation for the current I, where the flux $\Phi(t)$ through the ring is the driving parameter.
3. Identify the generalized susceptibility, and observe that it is formally the same expression as in the problem of Brownian motion.
4. Calculate the current-current correlation function $\langle I(0)I(\tau)\rangle$ taking the classical / high temperature limit.
5. Calculate the current fluctuation $\left\langle I^2\right\rangle$,Verify that $\left\langle I^2\right\rangle$ agree with the canonical result.

Solution

1. $L = \mu_0 R\left[ln\left(\frac{8R}{r}\right) - 1.75\right]$.
2. Let us write equation for the force by Using $V_{EMF} = \dot{\Phi}$ we get

   $$RI + L\dot{I} = \dot{\Phi}$$

   By rearrange the equation and taking Transform Fourier we get

   $$I_w = \frac{i\omega}{R - i\omega L}\Phi_\omega$$

3. From the theory of liner respond we Identify the generalized susceptibility

   $$\chi_\omega = \frac{i\omega}{R - i\omega L}$$

   For Brownian motion the susceptibility is $\chi_\omega = \frac{i\omega}{\eta - i\omega M}$ and we get the same expression.

4. First we calculate $\Im(\chi_\omega)$

   $$\chi_\omega = \frac{-\omega^2 L}{R^2 + \omega^2 L^2} + i\frac{\omega R}{R^2 + \omega^2 L^2}$$

   $$\Im(\chi_\omega) = \frac{\omega R}{R^2 + \omega^2 L^2}$$

$$C_{II}^{Q}(\omega) = \hbar \cdot coth(\frac{\hbar\omega}{2K_BT})\Im(\chi_\omega)$$
$$C_{II}^{classic}(\omega) = \lim_{\hbar\to 0} C_{II}^{Q}(\omega) \longrightarrow \frac{2K_BTR}{R^2+\omega^2L^2}$$
$$\langle I(0)I(\tau)\rangle = \int_{-\infty}^{\infty} e^{-i\omega\tau} C_{II}^{classic}(\omega)\,\frac{d\omega}{2\pi}$$
$$\langle I(0)I(\tau)\rangle = \frac{K_BT}{L}e^{-\frac{R\tau}{L}}$$

In the case of Brownian motion we obtain the same expression $\frac{K_BT}{M}e^{-\frac{\eta\tau}{M}}$

5. Now we calculate the current fluctuation by taking $\tau \to 0$ to obtain

$$\left\langle I^2\right\rangle = \frac{K_BT}{L}$$

The Hamiltonian is $\mathcal{H} = \frac{1}{2}LI^2$ by using equal division rule for each quadratic term in the Hamiltonian we get the same result

$$\left\langle \frac{1}{2}LI^2\right\rangle = \frac{K_BT}{2}$$

and therefore

$$\left\langle I^2\right\rangle = \frac{K_BT}{L}$$