Ex 7005: Shot noise

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The problem:

The discreteness of the electron charge e implies that the current is not uniform in time and is a source of noise. Consider a vacuum tube in which electrons are emitted from the negative electrode and flow to the positive electrode; the probability of emitting any one electron is independent of when other electrons are emitted. Suppose that the current meter has a response time τ. If T_e is the average time between the emission of two electrons, then the average current is $\langle I\rangle = e/T_e = \frac{e}{\tau}\eta$, where $\eta = \tau/T_e$ is the transmission probability, $0 \leq \eta \leq 1$.

(a) Show that the fluctuations in I are $\langle(\Delta I)^2\rangle = \frac{e^2}{\tau^2}\eta(1-\eta)$. Why would you expect the fluctuations to vanish at both $\eta = 0$ and $\eta = 1$? [Hint: For each τ interval n_i is the number of electrons hitting the positive electrode. Therefore, it can be equal to $n_i = 0$ or $n_i = 1$ which results in an average $\langle n_i\rangle = \tau/T_e$; discretize time in units of τ.]

(b) Consider the meter response to be in the range $0 < |\omega| < 2\pi/\tau$. Show that for $\eta \ll 1$ the fluctuations in the frequency domain are $\langle(\Delta I)^2\rangle = \mathrm{e}\langle I\rangle$. What is the condition for this noise to dominate over the Johnson-Nyquist noise in the circuit?

(c) Show that the 3rd order commulant is $\langle(I-\langle I\rangle)^3\rangle = \frac{e^3}{\tau^3}\eta(1-\eta)(1-2\eta)$.

The solution:

(a) During the response time τ there is a probability of $\frac{\tau}{T}$ that the electron will hit the cathode. The average of n_i is then

$$\langle n_i\rangle = 1\cdot\frac{\tau}{T_e} + 0\cdot\left(1-\frac{\tau}{T_e}\right) = \frac{\tau}{T_e} \equiv \eta \tag{1}$$

The current is simply $I = e\frac{n_i}{\tau}$ .The average current is $\langle I\rangle = \frac{e}{\tau}\langle n_i\rangle = \frac{e}{\tau}\eta$.

The fluctuations in I are:

$$\langle(\Delta I)^2\rangle = \langle I^2\rangle - \langle I\rangle^2 = \frac{e^2}{\tau^2}\langle n_i^2\rangle - \frac{e^2}{\tau^2}\eta^2 = \frac{e^2}{\tau^2}\eta\,(1-\eta) \tag{2}$$

where $\langle n_i^k\rangle = \langle n_i\rangle = \eta$ for $k \in \mathbb{N}$.

In the case where $\eta = 0$ or $\eta = 1$, there are no fluctuations, since the average current equals the minimum and maximum values that can be attained by I.

(b) The power spectrum $\tilde{C}(\omega)$ is the Fourier transform of $C(t)$, where $t = t_2 - t_1$ is the correlation time. Since the particles arrive at random times, the pulses in the current are uncorrelated. Therefore, $C(t)$ is of the form

$$C(t) = \langle\Delta I(t)\,\Delta I(0)\rangle = \langle(\Delta I^2)\rangle\delta(t) \tag{3}$$

Calculating $\tilde{C}(\omega)$ we get,

$$\tilde{C}(\omega) = \int_{-\infty}^{\infty} dte^{i\omega t}\langle(\Delta I^2)\rangle\delta(t) = \langle(\Delta I^2)\rangle = \frac{e^2}{\tau^2}\eta(1-\eta) \tag{4}$$

We get the fluctuations in the frequency domain from "Wiener-Khinchin theorem".

$$\langle|\Delta I_\omega|^2\rangle = \tilde{C}(\omega) \times \tau = \frac{e^2}{\tau}\eta(1-\eta) = e\langle I\rangle(1-\eta) \tag{5}$$

where τ is the averaging time. Also we used $\langle I\rangle = e\eta/\tau$ from previous result.
For $\eta \ll 1$ we finally receive

$$\langle|\Delta I_\omega|^2\rangle = e\langle I\rangle \tag{6}$$

This is the shot noise. This noise is white since it does not depend on the frequency.

The Johnson-Nyquist noise is a thermal noise in a circuit, with power spectrum equals to $2T/R$ in equilibrium, where T is the temperature and R is the resistance. In non-equilibrium conditions we might have the addition of a shot noise to the thermal noise. Decreasing the temperature, the shot noise becomes more important. At sufficiently low temperature the shot noise dominates the Johnson-Nyquist noise, $e\langle I\rangle > 2T/R$, which implies the condition

$$e\langle I\rangle R > 2T \tag{7}$$

(c) The 3rd order commulant is:

$$\langle(I-\langle I\rangle)^3\rangle = \frac{e^3}{\tau^3}(\langle n_i^3\rangle - 3\langle n_i^2\rangle\langle n_i\rangle + 3\langle n_i\rangle\langle n_i\rangle^2 - \langle n_i\rangle^3 = \frac{e^3}{\tau^3}\left(\eta - 3\eta^2 + 2\eta^3\right) \tag{8}$$

$$\langle(I-\langle I\rangle)^3\rangle = \frac{e^3}{\tau^3}\eta(1-\eta)(1-2\eta) \tag{9}$$