Exercises in Statistical Mechanics

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This exercises pool is intended for a graduate course in "statistical mechanics". Some of the problems are original, while other were assembled from various undocumented sources. In particular some problems originate from exams that were written by B. Horovitz (BGU), S. Fishman (Technion), and D. Cohen (BGU).

[Exercise 6776]

Boltzmann equation: Emission

Equilibrium and kinetics of light and matter:

(a) Consider atoms with fixed positions that can be either in their ground state a_0, or in an excited state a_1, which has a higher energy ϵ. If n_0 and n_1 are the densities of atoms in the the two levels, find the ratio n_1/n_0 at temperature T.

(b) Consider photons γ of frequency $\omega = \epsilon/\hbar$ and momentum $|\mathbf{p}| = \hbar\omega/c$, which can interact with the atoms through the following processes:

(i) *Spontaneous emission*: $a_1 \rightarrow a_0 + \gamma$

(ii) *Absorption*: $a_0 + \gamma \rightarrow a_1$

(iii) *Stimulated emission*: $a_1 + \gamma \rightarrow a_0 + \gamma + \gamma$.

Assume that spontaneous emission occurs with a probability σ_1 (per unit time and per unit (momentum)^3) and that absorption and stimulated emission have constant (angle independent) differential cross-sections of σ_2 and $\sigma_3/4\pi$, respectively.

Show that the Boltzmann equation for the density $f(\mathbf{r}, p, t)$ of the photon gas, treating the atoms as fixed scatterers of densities n_0 and n_1 is

$$\frac{\partial f(\mathbf{r},p,t)}{\partial t} + \frac{\mathbf{p}c}{|\mathbf{p}|} \cdot \frac{\partial f(\mathbf{r},p,t)}{\partial \mathbf{r}} = -\sigma_2 n_0 c f(\mathbf{r},p,t) + \sigma_3 n_1 c f(\mathbf{r},p,t) + \sigma_1 n_1$$

(c) Find the equilibrium solution f_{eq}. Equate the result, using (a), to that the expected value per state $f_{eq} = \frac{1}{h^3}\frac{1}{e^{\hbar\omega/k_B T}-1}$ and deduce relations between the cross sections.

(d) Consider a situation in which light shines along the x axis on a collection of atoms whose boundary is at $x = 0$ (see figure). The incoming flux is uniform and has photons of momentum $\mathbf{p} = \hbar\omega\hat{x}/c$ where $\hat{x}$ is a unit vector in the x direction. Show that the solution has the form

$$Ae^{-x/a} + f_{eq}$$

and find the penetration length a.