E6050: Radiation from 1D balckbody fiber

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The problem:
Given an optical fiber with length L and it's section area is A. The fiber is in thermic equilibrium at temperature T. Assume the fiber is a one dimensional medium for the electromagnetic field (relate to the magnetic radiation as a photon gas and the medium is like vacuum).

(a) What is the electromagnetic energy density per length unit?

(b) What is the radiation pressure on the fiber edges?

(c) Assuming that the radiation can emit from the boundary of the fiber, what is the emitting radiation flow (energy per time unit)?

(d) What is the spectral distribution $J(w)$ of the emitting radiation flow?

(e) What is the entropy and what is the heat capacity of the system?

Given:

$$\int_0^\infty \frac{x}{e^x - 1} dx = \frac{\pi^2}{6}$$

Treat the photon gas in the medium like a standing wave.

The solution:

(a) If we consider the optical fiber like a cavity filled with radiation in thermodynamic equilibrium, and it's purpose is to confine the radiation long enough for it to reach equilibrium. So the walls of the optical fiber must be with nonzero conductivity, otherwise it would be transparent. Therefore in equilibrium the electric field strength at the walls is $E = 0$. This means that we have standings waves.

Moreover, In order to calculate the electromagnetic energy density per length unit, we will first have to calculate the number of modes. In of oscillation of electromagnetic radiation possible in a cavity, in our case it is a one dimensional optical fiber of length $L/$.As mention above, in equilibrium only standing waves are possible, and these will have nodes at the ends $x = 0, L$.

$$L = \frac{n_x}{2}\lambda; n_x = 0, 1, 2, ...$$

Another important factor is the polarization of the electromagnetic wave. In a standing electromagnetic wave in the cavity, just as in a propagating wave, the electric field is perpendicular to the direction in which the wave is moving. In other words, it is a transverse wave, and in fact there are two independent polarizations, which we must remember to count when we find the total number of degrees of freedom.

For electromagnetic waves in vacuum, the dispersion relation is $\omega = ck$. So the angular frequency is now quantized. The symbol c here is the speed of light.

$$\omega = c\frac{\pi \cdot n_x}{L}$$

The permitted standing wave modes can be represented as a lattice of straight line of points n_x. The number of modes untile N is $N = 2n = \frac{2L\omega}{c\pi}$. Note that the factor of 2 representing the two independent polarizations of the electromagnetic wave. Hence the density of modes is

$$g(\omega) = \frac{2L}{c\pi}$$

Its worth to mention that there can be another way to drive this density of states. If we will let the cavity become infinitely large, so the boundary conditions and shape will not matter, then we can use periodic boundary conditions, which state that the plane waves are equal on opposite boundaries

$$e^{ik\cdot x} = e^{ik\cdot(x+L)} \Rightarrow e^{ik\cdot L} = 1 \Rightarrow k = \frac{2\pi n}{L}$$

For all possible values of k. So the density of states is therefore,

$$g(\omega) = 2 \cdot 2 \cdot \frac{L}{2\pi c} = \frac{2L}{c\pi}$$

The first "2" is for the polarizations, the second is for counting $\omega = c\,|k|$.

If we look at the electromagnetic field inside the optical fiber as a gas of indentical and indistinguishable quanta (photons), then the energy of a photon is $\epsilon_s = \hbar\omega_s n$ where $n = 0, 1, 2, ...$The oscillators are distinguisable from one another by their value of ω_s, hence they obey Maxwell-Boltzmann statistics. So the average energy of frequency ω_n is then

$$< \epsilon_s >= \frac{\sum_{n=0}^{\infty} \epsilon_n e^{-\frac{\epsilon_n}{kT}}}{\sum_{n=0}^{\infty} e^{-\frac{\epsilon_n}{kT}}} = \frac{\hbar\omega_s}{e^{\frac{\hbar\omega_s}{kT}} - 1}$$

Now the total energy per unit length is the integral over all frequencies

$$\frac{U}{L} = \int_0^{\infty} g(\omega_s) < \epsilon_s > d\omega = \frac{2}{\pi c}\int_0^{\infty} \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1} d\omega$$

Ssubstituting ω by $\omega = \frac{kT}{\hbar}x$

$$\frac{U}{L} = \frac{2k^2T^2}{\pi c\hbar}\int_0^{\infty} \frac{xdx}{e^x - 1} = \frac{\pi k^2 T^2}{3c\hbar}$$

(b) From the fundamental thermodynamic relation $dU = TdS - PdL$ we can conclude that the radiation pressure on the fiber edges is equal to the following

$$U = \frac{\pi k^2 T^2}{3c\hbar} L \Rightarrow p = -\frac{\pi k^2 T^2}{3c\hbar}$$

(c) The radiation flux is said to be emitted from only the two said of the optical fiberm. So there is a factor of half, if we look only on the flux from the right said alone. By definition the radiation flow has a units of energy per time unit per area unit. In one dimenstion at time interval of Δt the photons have traveled a distance of $c\Delta t$.

$$J_{1D} = \frac{1}{2} \cdot \frac{U}{V} \cdot \frac{Ac\Delta t}{A\Delta t} = \frac{1}{2}\frac{U}{V}c = \frac{\pi k^2 T^2}{6\hbar}$$

(d) If we want to calculate the spectral distribution $J('omega)$ of the emitting radiation flow then we first have to calculate the average number of photons with frequency ω_s

$$< N_\omega >= \sum_s p(s) \cdot s = \frac{1}{Z} \sum_s s e^{-\beta\hbar\omega_s} = \frac{1}{e^{\beta\hbar\omega} - 1}$$

Then the energy of the system in mode ω_s is

$$< \epsilon_s >= \hbar\omega < N_s >= \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}$$

We must remember to include the density of states, hence the radiation flow is

$$J_{1D}(\omega) = \frac{1}{2} c \cdot g(\omega) \cdot \frac{< \epsilon_\omega >}{L} = \frac{1}{2} c \cdot \frac{2L}{\pi c} \cdot \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} \cdot \frac{1}{L} = \frac{1}{\pi} \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}$$

(e) As we said before the from the energy equation

$$\Omega = -L \cdot Pressure = p = -L \cdot (-\frac{\pi k^2 T^2}{3c\hbar}) = \frac{L\pi k^2 T^2}{3c\hbar}$$

Moreover, we can calculate the entropy and the heat capacity of the system by using the following equations

$$S = -\frac{\partial \Omega}{\partial T} = 2\frac{L\pi k^2 T}{3c\hbar}$$

$$c = T\frac{\partial S}{\partial T} = 2\frac{L\pi k^2 T}{3c\hbar}$$