Ex6030: Thermionic emission of electrons from a metal

Submitted by: Gideon Ilani

The problem:

A spherical piece of metal (cathode), that has radius R and temperature T , is placed inside a vacuum tube. A second metallic piece (anode) is used to collect the electrons that are emitted from the cathode. The effective temperature of the anode is zero. The cathode has a work function W, while the anode has work function W'. The depth of the potential that holds the electrons inside the cathode, aka the potential floor, is V_0.

(1) Write an integral expression for the saturation current I_s that would be measured when the bias voltage is very large.
(1a) Show that V_0 does not appear in the final result: the outcome of the calculation is the same for sections that are close to the surface or deep in the metal.
(1b) Calculate the integral using the Boltzmann approximation. Specify the range of temperatures for which the approximation is valid.
(2) Using the result of the previous item write an estimate for the current if a reverse (stopping) voltage V_{battery} is applied. Explain whether W or W' is relevant.
(2a) Explain the relation to the analysis of the stopping voltage in the photoelectric effect.
(3) Assume that the cathode is detached and left alone in free space. Calculate the charge $Q(t)$ of the cathode as a function of time assuming that $Q(0) = 0$.
(3a) Explain the limitations of the result that you have obtained.

The solution:

(1) When the bias is very large every electron that manages to get out of the cathode to the vacuum with some velocity will get to the anode. The electrons in the cathode have energy $\epsilon = \frac{mv^2}{2} - eV_0$, and in the vacuum (without the bias) they have energy $\epsilon_{\text{vacuum}} = \frac{mv^2}{2}$. We can imagine that our surface is perpendicular to the z axis, so if the electrons are to leave the cathode with velocity we should demand that:

$$\frac{mv_{\text{z,vacuum}}^2}{2} = \frac{mv_{\text{z,cathode}}^2}{2} - eV_0 > 0 \tag{1}$$

which means that only electrons with velocity:

$$v_{\text{z}} > v_{\text{z,min}} = \left(\frac{2eV_0}{m}\right)^{\frac{1}{2}} \tag{2}$$

will contribute to the current. The current will be the current incident on a unit surface element multiplied by the surface area, so our integral expression is:

$$I_s = 4\pi R^2 J_s = 4\pi R^2 \int_{-\infty}^{\infty} d^2v \int_{v_{\text{z,min}}}^{\infty} v_z dv_z \; 2\left(\frac{m}{2\pi}\right)^3 f\left(\frac{mv^2}{2} - eV_0 - \mu\right) \tag{3}$$

where we have taken spin degeneracy into consideration, and $f\left(\frac{mv^2}{2} - eV_0 - \mu\right)$ is the Fermi occupation.

(1a) We can change variables as follows:

$$\frac{mv_z'^2}{2} = \frac{mv_z^2}{2} - eV_0, \; v_z' dv_z' = v_z dv_z, \; v_{\text{z,min}}' = v_{\text{z,min}} - \left(\frac{2eV_0}{m}\right)^{\frac{1}{2}} = 0 \tag{4}$$

This will give us:

$$I_s = 4\pi R^2 \int_{-\infty}^{\infty} d^2v \int_0^{\infty} dv_z' v_z' 2 \left(\frac{m}{2\pi}\right)^3 f\left(\frac{mv'^2}{2} - \mu\right) \tag{5}$$

This integral is independent of V_0. Physically this is due to detailed balance: In equilibrium, the net current is zero, so if there is a current into the cathode the current out of it will be determined by the demand that it will cancel the current into it. The current into the cathode from the vacuum is clearly not related to the cathode potential floor, so this is true also for the current out of the cathode.

(1b) In the Boltzmann approximation our integral becomes:

$$I_s = 8\pi^2 R^2 \int_0^1 \int_0^{\infty} 2\left(\frac{m}{2\pi}\right)^3 e^{-\beta\left(\frac{mv'^2}{2}+W\right)} v^3 \cos\theta \, d(\cos\theta)\, dv \tag{6}$$

Because W is defined as $W = e\phi_{\text{surface}} - \mu$, where ϕ_{surface} is the potential on the surface of the cathode, which we can choose to be 0, then $\mu = -W$.
Changing variables to get dimensionless variables:

$$I_s = 8R^2T^2e^{-\beta W}\left(\frac{m}{\pi}\right)\int_0^{\infty} e^{-x^2}x^3 dx = \left(\frac{4mR^2}{\pi}\right)T^2e^{-\beta W} \tag{7}$$

For the Boltzmann approximation to be valid we must demand that for the lowest energy (here - $mv'^2_{\text{min}}/2 = 0$) $e^{\beta(\epsilon_{\text{min}}-\mu)} \gg 1$, which means that $-\frac{\mu}{T} \gg 1$, or $\mu \ll -T$.

(2) In the above relation we see that the thermionic emission here is proportional to $T^2e^{\beta W}$. If there is no stopping voltage the energetic barrier an electron has to pass is W. When there is stopping voltage the energetic barrier becomes

$$\mu_{\text{anode}} - \mu_{\text{cathode}} + W' = W' + eV_{\text{battery}}, \tag{8}$$

so we expect that the expression for the current will be:

$$I \propto T^2 e^{-\beta\left(W' + eV_{\text{battery}}\right)} \tag{9}$$

We used $\propto$ because our argument does not account for the geometric prefactors.

(2a) In the photoelectric effect the cathode isn't heated so one assumes zero temperature Fermi occupation for the cathode also. That means there is no thermionic emission. The electrons are excited out of the cathode by photons with a fixed energy. These photons need to be with enough energy for the electrons to pass the energetic barrier, which is again $W' + eV_{\text{battery}}$. So if we measure the voltage needed to stop the current for a given photons energy, we know that at that voltage we have:

$$E_{\text{photon}} = W' + eV_{\text{battery}} \implies W' = E_{\text{photon}} - eV_{\text{battery}} \tag{10}$$

So we can calculate the anode work function.

(3) We can treat the cathode as a capacitor. The charge will be $Q = VC$, and the current: $I = C\frac{d}{dt}V$. Here $C = \frac{1}{R}$, and V are the capacitance and voltage of a charged sphere in vacuum with respect to infinity. On the other hand we know from item (2) that when there is a voltage difference the current is $\sim I_0 e^{-\beta eV}$, where I_0 is independent of V. So we need to solve:

$$\frac{d}{dt}V = \frac{I_0}{C}e^{-\beta eV} \tag{11}$$

given that $Q\left(t=0\right)=0$:

$$e^{\beta eV}dV=\frac{I_0}{C}dt\implies\frac{T}{e}e^{\beta eV}=\frac{I_0}{C}t+\frac{T}{e} \tag{12}$$

$$V=\frac{T}{e}\ln\left(\frac{e}{T}\frac{I_0}{C}t+1\right)\implies Q\left(t\right)=\frac{CT}{e}\ln\left(\frac{e}{T}\frac{I_0}{C}t+1\right) \tag{13}$$

(3a) The result is limited because it is valid only when the Boltzmann approximation is valid, and it does not take into consideration finite number of electrons.