Ex 6020: A divided box with a hole in one side

Submitted by: Yotam Sherf

The problem:

A cylinder of length L and cross section A is divided into two compartments by a piston. The piston has mass M and it is free to move without friction. Its distance from the left basis of the cylinder is denoted by x. In the left side of the piston there is an ideal Bose gas of N_a particles with mass m_a. In the right side of the piston there is an ideal Bose gas of N_b particles with mass m_b. The temperature of the system is T. Assume that the left gas can be treated within the framework of the Boltzmann approximation. Assume that the right gas is in condensation. In items (3-5) consider separately two cases:

(a) A small hole is drilled in the left wall of the box.
(b) A small hole is drilled in the right wall of the box.

The area of the hole is δA.

(1) Find the equilibrium position of the piston.

(2) What is the frequency of small oscillations of the piston.

(3) What is the velocity distribution $F(v)$ of the emitted particles?

(4) What is the flux (particles per unit time) of the emitted particles?

(5) Is the piston going to move? If yes write an expression for its velocity.

In item (3) use normalization that makes sense for the calculation in item (4). In item (5) assume that the process is quasi-static, such that at any moment the system is at equilibrium. Express your answers using $L, A, \delta A, N_a, N_b, \mathsf{m}_a, \mathsf{m}_b, T, M$.

$$\int_0^\infty \frac{xdx}{e^x - 1} = \frac{\pi^2}{6}$$

The solution:

(1)

Equilibrium condition

$$P_a = P_b \tag{1}$$

Where for Boltzmann $P_a = \dfrac{N_a T}{Ax}$ and for the condense gas $Li_{5/2}(1) = \zeta(5/2)$ and $P_b = \zeta\left(\frac{5}{2}\right)\left(\frac{m_b}{2\pi}\right)^{\frac{3}{2}} T^{\frac{5}{2}}$.

Define x_0 as the equilibrium position we find from (1).

$$x_0 = \frac{N_a}{\zeta(5/2)A}\left(\frac{2\pi}{m_b T}\right)^{3/2} \tag{2}$$

(2)

Looking at the total force on the piston and expanding it around x_0 for small oscillations.

$$F(x_0+dx) = F(x_0) + \underbrace{\frac{dF(x)}{dx}}_{-Mw^2}\Bigg|_{x_0} dx = A\Bigg(P(x_0) + \frac{dP(x)}{dx}\Bigg|_{x_0} dx\Bigg) \tag{3}$$

We find

$$w^2 = -\frac{A}{M}P'(x_0) = \frac{N_a T}{M x_0^2} \quad \Rightarrow \quad w = \zeta(5/2)\frac{A}{\sqrt{N_a M}}\left(\frac{m_b}{2\pi}\right)^{3/2} T^2 \tag{4}$$

(3)

(a) In the Boltzmann case we have the known velocity distribution

$$n(v) = \left(\frac{m_a}{2\pi T}\right)^{3/2} 4\pi v^2 e^{-\frac{\beta m v_a^2}{2}} \tag{5}$$

And for N_a particles we have

$$F_a(v) = N_a n(v) = N_a\left(\frac{m_a}{2\pi T}\right)^{3/2} 4\pi v^2 e^{-\frac{\beta m_a v^2}{2}} \tag{6}$$

(b)condense Bose gas.

$$F_b(v)dv = \frac{1}{(2\pi)^3}\left(\int d^3x\right) m_B^3 d^3v \frac{1}{e^{-\frac{\beta m_B v^2}{2}} - 1} = V\left(\frac{m_b}{2\pi}\right)^3 4\pi v^2 \frac{1}{e^{-\frac{\beta m_b v^2}{2}} - 1} dv \tag{7}$$

Where $V = A(L-x)$.

(4)

By definition $J(v) = \rho(v)v$, and $\rho(v) = \dfrac{F(v)}{V}$, so the total flux.

(a) Boltzmann gas.

$$\begin{aligned} J_a &= 4\pi\frac{N_a}{4V}\left(\frac{m_a}{2\pi T}\right)^{3/2}\int_0^\infty v^3 e^{-\frac{\beta m_B v^2}{2}} dv = \frac{2N_a\pi}{Ax_0}\left(\frac{m_a}{\pi T}\right)^{3/2}\left(\frac{2T}{m_a}\right)^2\int_0^\infty x e^{-x} dx = \\ &\zeta(5/2)\left(\frac{T}{2\pi}\right)^2\sqrt{\frac{m_b^3}{m_a}}\underbrace{\int_0^\infty x e^{-x} dx}_{1} = \zeta(5/2)\left(\frac{T}{2\pi}\right)^2\sqrt{\frac{m_b^3}{m_a}} \end{aligned} \tag{8}$$

And the flux per unit time $I_a = \delta A J_a$

(b) Bose gas.

$$I_b = (\delta A)2\pi\left(\frac{m_b}{2\pi}\right)^3\left(\frac{T}{m_b}\right)^2\underbrace{\int_0^\infty \frac{x}{e^{-x}-1}dx}_{\pi^2/6} = (\delta A)\frac{m_b}{6}\left(\frac{T}{2}\right)^2 \tag{9}$$

with no dimensional dependence.

(5)

(a) In the Boltzmann case using the quasi-stationary assumption we can use equilibrium condition (1) in each moment, so $x_0 \to x_0(t)$.

$$x_0(t) = \frac{2\pi}{A\zeta(5/2)} \left(\frac{2\pi}{m_b T}\right)^{3/2} N_a(t) \tag{10}$$

where $\dfrac{dN_a(t)}{dt} = I_a$, the piston velocity would be ($\dot{x}_0(t) = v_p$).

$$v_p = \frac{\delta A}{A} \sqrt{\frac{2\pi T}{m_a}} \tag{11}$$

(b) In the condense case we have $P_b = \zeta\left(\frac{5}{2}\right)\left(\frac{m_b}{2\pi}\right)^{\frac{3}{2}} T^{\frac{5}{2}}$ with no dependence on N_b, so the pressure will remain the same and the equilibrium condition (1) still holds .