Ex6010: Effusion from a box with Bose gas and magnetic field

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The problem:

A condensed Bose gas with spin 1, mass m and gyromagnetic ratio γ is placed in a box with temperature T and a strong magnetic field B. A δA area hole is drilled in the box so the gas can effuse. The effused particle flux is separated into 3 beams using a Stern-Gerlach machine. Each beam is directed into a different container.

(1) Write the single particle Hamiltonian.
(2) Find the velocity distribution $F_{S_z}(v)$ for $S_z = -1, 0, 1$.
(3) Define what is a strong magnetic field, and explain why and how it helps for the solution of the next item.
(4) Find how many particles are accumulated in each container after time t.
(5) Find what would be the velocity distribution for horizontal filtering S_x of the spin.
Express your answer with $B, \delta A, t, \gamma, m, T$. In the last item use the $F_{S_z}(v)$ as given, even if item2 has not been solved.

You may find the following useful:

$$\int_0^\infty x^3 e^{-x^2} dx = \frac{1}{2}, \int_0^\infty \frac{x^3}{e^{x^2}-1} dx = \frac{\pi^2}{12}$$

The solution:

(1)

$$\epsilon = \frac{P^2}{2m} - \gamma B S_z$$

(2) First, we need to find the density of states function for each spin:

$$N_{states} = \int\int \frac{d^3x d^3p}{(2\pi)^3} = \frac{V}{(2\pi)^3} 4\pi \int p^2 dp = \frac{V}{(2\pi)^3} 4\pi \int_{-\gamma B S_z}^{\epsilon} 2m(\epsilon + \gamma B S_z) \frac{1}{2} \sqrt{\frac{2m}{\epsilon + \gamma B S_z}} d\epsilon$$

$$g_{s_z}(\epsilon) = \frac{dN_{states}(\epsilon)}{d\epsilon} = 2\pi \frac{V}{(2\pi)^3} (2m)^{3/2} (\epsilon + \gamma B S_z)^{1/2}$$

In order to find the velocity distribution we will change the variable from ϵ to v. Now, the density of states per small velocity interval is:

$$g(v) = 2\pi \frac{V}{(2\pi)^3} (2m)^{3/2} (\frac{1}{2} m v^2)^{1/2}$$

And the average particles at a velocity state is:

$$f(v) = \frac{1}{e^{\beta(\frac{1}{2} m v^2 - \gamma B S_z - \mu)}}$$

Hence, the velocity distribution is (compare with 751 in DC lecture notes)

$$F_{S_z}(v)dv = g(v)f(v)d(\frac{1}{2}mv^2 - \gamma BS_z) = 2\pi\frac{V}{(2\pi)^3}(2m)^{3/2}(\frac{1}{2}mv^2)^{1/2}\frac{1}{e^{\beta(\frac{1}{2}mv^2-\gamma BS_z-\mu)}-1}mvdv$$

$$F_{S_z}(v)dv = V \times 4\pi v^2(\frac{m}{2\pi})^3\frac{1}{e^{\beta(\frac{1}{2}mv^2-\gamma BS_z-\mu)}-1}dv$$

The gas is condensed, so by putting $\mu = -\gamma B$ we find:

$$F_1(v)dv = V \times 4\pi v^2(\frac{m}{2\pi})^3\frac{1}{e^{\beta\frac{1}{2}mv^2}-1}dv$$

$$F_0(v)dv = V \times 4\pi v^2(\frac{m}{2\pi})^3\frac{1}{e^{\beta(\frac{1}{2}mv^2+\gamma B)}-1}dv$$

$$F_{-1}(v)dv = V \times 4\pi v^2(\frac{m}{2\pi})^3\frac{1}{e^{\beta(\frac{1}{2}mv^2+2\gamma B)}-1}dv$$

(3) the total flux is:

$$J_{S_z} = \int_0^\infty \frac{1}{4}\frac{F_{S_z}(v)dv}{V}v$$

And the number of particles that are accumulated in each container after time t is:

$$N_{S_z} = J_{S_z}\delta At$$

By assuming a strong magnetic field relative to the temperature $\gamma B >> T$, which cause the $s = 0$ and $s = -1$ population to behave like an ideal gas, we can rewrite the probability function, $f(\epsilon - \mu)$ for $S = o, -1$:

$$f_0 = \frac{1}{e^{\beta\frac{1}{2}mv^2+\beta\gamma B}-1} \simeq e^{-\beta(\frac{1}{2}mv^2+\gamma B)}$$

$$f_{-1} = \frac{1}{e^{\beta\frac{1}{2}mv^2+2\beta\gamma B}-1} \simeq e^{-\beta(\frac{1}{2}mv^2+2\gamma B)}$$

which is much more easier to integrate.

(4) Using the given integrals gives us the desired answer:
The flux for $S_z = 1$ is:

$$J_1 = \int_0^\infty \frac{1}{4}\frac{F_1(v)dv}{V}v = \int_0^\infty \pi v^2(\frac{m}{2\pi})^3\frac{v}{e^{\beta\frac{1}{2}mv^2}-1}dv = \{x^2 = \beta\frac{1}{2}mv^2\} =$$

$$= \int_0^\infty \pi(\frac{m}{2\pi})^3\frac{x^3}{e^{x^2}-1}(\frac{2T}{m})^2dx = \frac{mT^2}{2\pi^2}\int_0^\infty \frac{x^3}{e^{x^2}-1}dx = \frac{mT^2}{24}$$

Therefor the number of particles in $S_z = 1$ container is:

$$N_1 = \frac{mT^2}{24}\delta At$$

The flux for $S_z = 0$ is:

$$J_0 = \int_0^\infty \frac{1}{4}\frac{F_0(v)dv}{V}v = \int_0^\infty \pi v^2 (\frac{m}{2\pi})^3 \frac{v}{e^{\beta(\frac{1}{2}mv^2+\gamma B)}-1}dv \simeq \{x^2 = \beta\frac{1}{2}mv^2\} \simeq$$

$$= \int_0^\infty \pi(\frac{m}{2\pi})^3 x^3 (\frac{2T}{m})^2 e^{x^2-\gamma B}dx = \frac{mT^2}{2\pi^2}\int_0^\infty x^3 e^{-x^2-\gamma B}dx = \frac{mT^2}{4\pi^2}e^{-\gamma B}$$

Therefor the number of particles in $S_z = 0$ container is:

$$N_0 = \frac{mT^2}{4\pi^2}\delta Ate^{-\gamma B}$$

The flux for $S_z = -1$ is:

$$J_{-1} = \int_0^\infty \frac{1}{4}\frac{F_0(v)dv}{V}v = \int_0^\infty V \times 4\pi v^2 (\frac{m}{2\pi})^3 \frac{v}{e^{\beta(\frac{1}{2}mv^2+2\gamma B)}-1}dv \simeq \{x^2 = \beta\frac{1}{2}mv^2\} =$$

$$= \pi(\frac{m}{2\pi})^3 x^3 (\frac{2T}{m})^2 x^3 e^{-x^2-2\gamma B}dx = \frac{mT^2}{2\pi^2}\int_0^\infty x^3 e^{-x^2-2\gamma B}dx = \frac{mT^2}{4\pi^2}e^{-2\gamma B}$$

Therefor the number of particles in $S_z = -1$ container is:

$$N_{-1} = \frac{mT^2}{4\pi^2}\delta Ate^{-2\gamma B}$$

(5) The filter tells as the percent of particles that are going through him, so we can think on the filters (x or z) as if they represent the probability distribution. And since we know the probability relations between S_x and S_z (equation 422 in DC QM lecture notes) we can write the $F_{S_x}(v)$ without knowing the exact form of $F_{S_z}(v)$:

$$S_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$|S_x = 1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, |S_x = -1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}, |S_x = 0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

hence,

$$F_{S_x=1} = \frac{1}{4}(F_1 + 2F_0 + F_{-1})$$

$$F_{S_x=-1} = \frac{1}{4}(F_1 + 2F_0 + F_{-1})$$

$$F_{S_x=0} = \frac{1}{2}(F_1 + F_{-1})$$