Exercises in Statistical Mechanics

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This exercises pool is intended for a graduate course in "statistical mechanics". Some of the problems are original, while other were assembled from various undocumented sources. In particular some problems originate from exams that were written by B. Horovitz (BGU), S. Fishman (Technion), and D. Cohen (BGU).

====== [Exercise 5980]

BEC regarded as a phase transition

Consider N bosons that each have mass M in a box of volume V. The overall density of the particles is $\rho = N/V$. The temperature is T. Denote by m the number of particles that occupy the ground state orbital of the box. The canonical partition function of the system can be written as

$$Z \;=\; \sum_{m=0}^{N} Z_{N-m} \;=\; \sum_{m=0}^{N} e^{-\tilde{A}(m)} \;=\; \int d\varphi \; e^{-N\,A(\varphi)+\text{const}}$$

In this question you are requested to regard the the Bose-Einstein condensation as phase transition that can be handled within the framework of the canonical formalism where m is the order parameter. Whenever approximations are required assume that $1 \ll m \ll N$ such that $\varphi = (m/N)$ can be treated as a continuous variable. In the first part of the question assume that the gas is ideal, and that Z_{N-m} can be calculated using the Gibbs prescription. In item 5 you are requested to take into account the interactions between the particles. Due to the interactions the dispersion relation in the presence of m condensed bosons is modified as follows:

$$E_k \;=\; \sqrt{\left(\epsilon_k + 2g\frac{m}{V}\right)\epsilon_k}$$

where ϵ_k are the one-particle energies in the absence of interaction, and g is the interaction strength. For the purpose of evaluating Z_{N-m} for large m assume that the above dispersion relation can be approximated by a linear function $E_k \propto k$

(1) Write an explicit expression for the probability p_m of finding m particles in the ground state orbital. Calculation of the overall normalization factor is not required.

(2) Find the most probable value $\bar{m}$. Determine what is the condensation temperature T_c below which the result is non-zero.

(3) Assuming $T < T_c$ write a Gaussian approximation for p_m

(4) Using the Gaussian approximation determine the dispersion δm

(5) Correct your answer for p_m in the large m range where the interactions dominate.

(6) On the basis of your answer to item3, write an expression for $A(\varphi; f)$ that involves a single parameter f whose definition should be provided using ρ, M, T.

(7) On the basis of your answer to item5, write an expression for $A(\varphi; a)$ that involves a single parameter a whose definition should be provided using ρ, M, T and g.

Tip: Only Stirling's approximation for $\ln((N-m)!)$ is requested/required/allowed in this question.