E5969: 2D Coulomb gas

Submitted by: Michael Epshtein

The problem:

N ions of positive charge q and N ions of negative charge -q are constrained to move in a two dimensional squre of side L and area L^2. the interaction energy of charge q_i at position r$_i$ with another charge q_j at position r$_j$ is $-q_i q_j \; ln\,|r_i - r_j|$ where $q_i, q_j = \pm q$. mass of the ions is m.

(a) By rescaling space variables to $r_i = \frac{1}{L} r_i$, The partition function can be written as $Z(L) = CL^\alpha$ find α where C dont depend on L. Note that: $\sum\limits_{i<j}^{2N} q_i q_j$has a simple dependence on N

(b) Calculate the pressure and show that for T<T$_c$ the system is unstable, determine what is T$_c$. Comment on the reason for this instability.

(c) Determine what is C if the interaction can be neglected, handle Gibbs correctly.

(d) Find the chemical potential $\mu(T, N, L)$ and solve for $N(\mu, T, L)$. In particular write what is N for a fixed μ in the limit $A \to \infty$ for both $T > T_c$ and $T < T_c$.

The solution:

(a) we can assume that $\sum\limits_{j=1}^{N} q_j = Nq$ and $\sum\limits_{j=N+1}^{2N} q_j = -Nq$

$$\sum_{i<j}^{2N} q_i q_j = \sum_{j=2}^{2N} q_1 q_j + \sum_{j=3}^{2N} q_2 q_j + ... + \sum_{j=N}^{2N} q_{N-1} q_j + \sum_{j=N+1}^{2N} q_N q_j + \sum_{j=N+2}^{2N} q_{N+1} q_j + ... + \sum_{j=2N}^{2N} q_{2N-1} q_j = \sum_{j=N+1}^{2N} q_N q_j = -Nq^2$$

For understanding this result let us look at $\sum\limits_{j=N}^{2N} q_{N-1} q_j$, here q_{N-1} is a positive charge that interact with N negative charges from q_{N+1} to q_{2N} and one positive charge $q_N$therefore
$\sum\limits_{j=N}^{2N} q_{N-1} q_j = -(N-1)q^2$

$\sum\limits_{j=N+2}^{2N} q_{N+1} q_j$ here q$_{N+1}$is a negative charge that interact with only negative charges from q_{N+2} to q_{2N} therefore $\sum\limits_{j=N+2}^{2N} q_{N+1} q_j = +(N-1)q^2$, all other parts will reduced in same way

only $\sum\limits_{j=N+1}^{2N} q_N q_j$ remains while all N ions in this summation are negative .

$$H = \sum_{i=1}^{2N} \frac{p_i^2}{2m} - \sum_{i<j}^{2N} q_i q_j \; ln\,|r_i - r_j|$$

$$Z_{(L)} = \prod_i^{2N} \int_{-\infty}^{+\infty} exp(-\beta \frac{p_i^2}{2m})\ dp_i \prod_i^{2N} \int_{L^2} exp\ (\beta \sum_{i<j}^{2N} q_1 q_j\ \ \ln|r_i - r_j|)\ dr_i = \frac{1}{\lambda^{4N}} \prod_i^{2N} \int_{L^2} exp\ (\beta \sum_{i<j}^{2N} q_1 q_j\ \ \ln|r_i - r_j|)\ dr_i$$

for $r_i = \frac{1}{L} r_i$

$$Z_{(L)} = \frac{1}{\lambda^{4N}} \prod_i^{2N} \int_0^1 exp\ (\beta \sum_{i<j}^{2N} q_i q_j\ \ \ln \frac{1}{L} |r_i - r_j|)\ (\frac{1}{L})^2 dr_i = \frac{1}{\lambda^{4N}}\ L^{\beta N q^2} \prod_i^{2N} \int exp\ (\beta \sum_{i<j}^{2N} q_i q_j\ \ \ln |r_i - r_j|)\ (\frac{1}{L})^2 dr_i$$

$$= C\ L^{(\beta N q^2 - 4N)}$$

Finelly we get $\alpha = \beta N q^2 - 4N$

(b) $P = \left(\frac{\partial F}{\partial L^2}\right)_{T,N} = \tau \frac{\partial}{\partial L^2} \ln Z(L) = \tau \frac{\partial}{\partial L^2} \ln\ (C\ L^{(4N - \beta N q^2)}) = \tau\ \frac{N}{L^2}\ (\ 2 - \frac{1}{2}\beta q^2\ )$

The system is unstable when $(\frac{\partial P}{\partial L^2})_{T,N} > 0$

$$\left(\frac{\partial P}{\partial L^2}\right)_{T,N} = \frac{N}{L^2}(\frac{1}{2}q^2 - 2\tau) > 0$$

we define low T by $k_B T < k_B T_c = \frac{1}{4} q^2$

the reason for the unstabillity is that in low T the ions behavior will depend mostly from their charges and not T therefore the distance between oposite charged ions will be shorter.

for $T \gg q^2$we get a pressure of ideal gas.

(c) C depend only from momentum integrals and from the Gibbs factors therefore

$$C = \frac{1}{\lambda^{4N}} (\frac{1}{N!})^2$$

(d) $Z(L) = \frac{1}{\lambda^{4N}} (\frac{1}{N!})^2 L^{2(2N - \frac{1}{2}\beta N q^2)}$

$$F = -\tau\ lnZ(L) = 2N\tau\ [\ \ln(\frac{\lambda^2 N}{L^2}) - 1\ ] + \frac{1}{2} N q^2\ \ \ln L^2$$

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{L^2, T} = 2\tau\ \ \ln(\frac{\lambda^2 N}{L^2}) + \frac{1}{2} q^2\ 2 \ln L$$

$$N = \frac{1}{\lambda^2} L^{2(1 - \frac{1}{4} q^2 \beta)} \exp(\frac{1}{2}\beta\mu)$$

for A$\rightarrow \infty$

$T > T_c$, $N \rightarrow \infty$ the system acting as ideal gas $PV = NTk_B$

$T < T_c$, $N \rightarrow 0$ the distance between the particles will go to zero ratio to the area.