C26 (modified)
HW 2009 Q.4

2D coulomb gas

$$\mathcal{H} = \sum_{i=1}^{2N} \frac{p_i^2}{2m} - \sum_{i<j}^{2N} q_i q_j \ln|\vec r_i - \vec r_j|$$

a) $$\left(\sum_{i=1}^{2N} q_i\right)^2 = 0 \leftarrow \text{charge neutrality}$$

$$= \sum_{i\neq j=1}^{2N} q_i q_j + \sum_{i=1}^{2N} q_i^2 = 2\sum_{i<j} q_i q_j + \underbrace{\sum q_i^2}_{=2Nq^2}$$

$$\sum_{i<j} q_i q_j = -\frac{1}{\cancel{2}}\cancel{2}Nq^2 = -Nq^2$$

$$Z(L) = Z_{kin} \prod_{i=1}^{2N} \int_0^L d^2 r_i \, e^{\beta \sum_{i<j} q_i q_j \ln|\vec r_i - \vec r_j|}$$

$$\begin{pmatrix} r_i' = c r_i \\ dr_i' = c\, dr_i \end{pmatrix} \qquad = Z_{kin} \prod_{i=1}^{2N} \int_0^{CL} \underbrace{d^2 r_i'}_{} \, e^{\beta \sum_{i<j} q_i q_j \ln|\vec r_i' - \vec r_j'|} \underbrace{e^{-\beta \sum_{i<j} q_i q_j \ln C}}_{= C^{\beta N q^2}}$$

$$Z(L) = Z_{kin} C^{-4N} C^{\beta q^2 N} Z(CL)$$

for $C = \frac{1}{L}$ we have , $A = L^2$

$$\boxed{Z(L) = L^{4N - \beta q^2 N} Z(1) \sim A^{N(2 - \frac{1}{2}\beta q^2)}}$$

b) $$P = -\left(\frac{\partial F}{\partial A}\right)_{T,N} = \frac{kTN}{A}\left(2 - \frac{1}{2}\beta q^2\right)$$

instability when $\left(\frac{\partial P}{\partial A}\right)_{T,N} > 0$ ie. when $2kT - \frac{1}{2}q^2 < 0$

or $\boxed{kT < \frac{q^2}{4}}$, at low T the charges pair into bound states. This clashes with the $\int_0^{\sim}$ integral and requires a short distance cutoff.

c) $$Z(1) \sim \underbrace{\left(\lambda^{-2}\right)^{2N}}_{\text{momentum}} \cdot \underbrace{\left(\frac{1}{N!}\right)^2}_{\text{Gibbs}} \rightarrow Z_N(A) \sim \frac{1}{(N!)^2 \lambda^{4N}} A^{N(2 - \frac{1}{2}\beta q^2)}$$

$$F = kT\left[2N \ln \lambda^2 + 2N \ln N - 2N - N\left(2 - \frac{1}{2}\beta q^2\right)\ln A\right] + \text{const (doesn't depend on } N)$$

$$\mu = \frac{\partial F}{\partial N} = k_B T\left[2\ln\lambda^2 + 2\ln N - \left(2 - \tfrac{1}{2}\beta q^2\right)\ln A\right]$$

$$e^{\beta\mu} = \lambda^4 N^2 A^{-2+\frac{1}{2}\beta q^2} \rightarrow \boxed{N = \frac{1}{\lambda^2} A^{1-\beta q^2/4} e^{\frac{1}{2}\beta\mu}}$$

as $A \to \infty$,

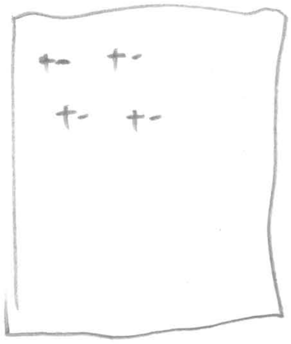
for $T > (T_c = q^2/4)$ we have $N \to \infty$ a gas of unbound particles.

for $T < T_c$ $N \to 0$

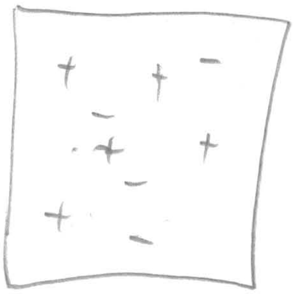
- Below T_c the system chooses to minimize the energy by allowing pairs to annhilate, leaving zero particles in the system. (ie. $|\bar{r}_i - \bar{r}_j| \to 0$)

When introducing the correct cutoff for non-pointlike particles, we get

Low $T < T_c$ High $T > T_c$



dipole gas



plasma