Exercises in Statistical Mechanics

Based on course by Doron Cohen, has to be proofed
Department of Physics, Ben-Gurion University, Beer-Sheva 84105, Israel

This exercises pool is intended for a graduate course in "statistical mechanics". Some of the problems are original, while other were assembled from various undocumented sources. In particular some problems originate from exams that were written by B. Horovitz (BGU), S. Fishman (Technion), and D. Cohen (BGU).

======= [Exercise 5969]

Baruch's C26.

N ions of positive charge q and N with negative charge $-q$ are constrained to move in a two dimensional square of side L. The interaction energy of charge q_i at position $\mathbf{r}_i$ with another charge q_j at $\mathbf{r}_j$ is $-q_i q_j \ln|\mathbf{r}_i - \mathbf{r}_j|$ where $q_i, q_j = \pm q$. The Hamiltonian is then (m is the mass of each ion and $\mathbf{p}_i$ are momenta)

$$\mathcal{H} = \sum_{i=1}^{2N} \mathbf{p}_i^2/2m - \sum_{i<j}^{2N} q_i q_j \ln|\mathbf{r}_i - \mathbf{r}_j|$$

(a) By rescaling space variables to $\mathbf{r}'_i = C\mathbf{r}_i$, where C is an arbitrary constant, show that the partition function $Z(L)$ satisfies: $Z(L) = C^{N(\beta q^2 - 4)} Z(CL)$. Deduce that $Z(L) = A^{N(2-\beta q^2/2)} Z(1)$ where $A = L^2$ is the area. [Hint: $\sum_{i<j}^{2N} q_i q_j = -q^2 N$].

(b) Calculate the pressure and show that at low T the system is unstable. Comment on the reason for this instability and on how the model should be modified.

(c) Assume that $Z(1)$ has N dependent factors only from the momentum integrals and from the Gibbs factors (this neglects a short range part of the interaction). Find the chemical potential $\mu(T, N, A)$ and solve for $N(\mu, T, A)$. Find the limit of N for a fixed μ when $A \to \infty$ for both $T > T_c = q^2/4$ and $T < T_c$. Interpret these results.