E5963: Stoner ferromagnetism

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The problem:

Consider Fermi gas of N spin $\frac{1}{2}$ electrons at temperature $T = 0$, with N_+ "up" electrons and N_- "down" electrons, such that $N = N_+ + N_-$. Due to the antisymmetry of the total wave function the energy of the system is:

$$U = \alpha \frac{N_+ N_-}{V}$$

Where V is the volume. Note that this interaction favors parallel spin states. Define the magnetization as $M = \frac{N_+ - N_-}{V}$.

(1) Write the total energy $E(M)$ including both the kinetic energy and the interaction, and expand up to 4th order in M.
(2) Find the critical value α_c such that for $\alpha > \alpha_c$ the electron gas can lower its total energy by spontaneously developing magnetization. This is known as the Stoner instability.
(3) Explain the instability qualitatively, and sketch the behavior of the spontaneous magnetization as a function of α.
(4) Repeat (a) at finite but low temperatures T, and find $\alpha_c(T)$ to second order in T.

The solution:

(1) The total energy is composed from the kinetic and potential energy $E_T = E_K + U$. We will first calculate the kinetic energy using prior knowledge of fermion gas and finally we will add the potential energy. First we note that at ground level the Fermi sea is completely populated and thus we can calculate for one of the spin directions

$$N_\pm = V \int_{k<k_F} \frac{d^3k}{(2\pi)^3} = V \int_0^{k_F} \frac{4\pi}{(2\pi)^3} k^2 dk = \frac{V k_{F\pm}^3}{6\pi^2} \Rightarrow k_{F\pm} = \left(6\pi^2 n_\pm\right)^{1/3}$$

Now lets evaluate the kinetic energy

$$E_K = 2V \int_{k_F} \epsilon(k) \frac{d^3k}{(2\pi)^3} = 2V \int_0^{k_F} \frac{\hbar^2 k^2}{2m} 4\pi k^2 dk = 2V \frac{\hbar^2}{2m} \frac{4\pi}{5(2\pi)^3} k_F^5$$

So for $T = 0$ the kinetic energy E_0 is

$$\frac{E_0}{V} = \frac{3}{5} \left(6\pi^2\right)^{2/3} \frac{\hbar^2}{2m} n^{5/3}$$

We wish to evaluate $n_+^{5/3}$ and $n_-^{5/3}$ at equilibrium therefore we take $n_+ = \frac{n}{2} + \delta$ and $n_- = \frac{n}{2} - \delta$ where $\delta << n$. The magnetization is $M = n_+ - n_- = 2\delta$. Let us expand $n_+^{5/3}$ to 4th order:

$$n_\pm^{5/3} = \left(\frac{n}{2}\right)^{\frac{5}{3}} \left(1 \pm \frac{2\delta}{n}\right)^{\frac{5}{3}} \cong \left(\frac{n}{2}\right)^{\frac{5}{3}} \left[1 \pm \frac{5}{3}x + \frac{\frac{5}{3}\frac{2}{3}}{2}x^2 \mp \frac{\frac{5}{3}\frac{2}{3}\frac{-1}{3}}{6}x^3 + \frac{\frac{5}{3}\frac{2}{3}\frac{-1}{3}\frac{-4}{3}}{24}x^4\right]$$

$$n_\pm^{5/3} = \left(\frac{n}{2}\right)^{\frac{5}{3}} \left[1 \pm \frac{5}{3}x + \frac{5}{9}x^2 \mp \frac{5}{81}x^3 + \frac{5}{243}x^4 + O(x^5)\right]$$

Where we denoted $x = \frac{2\delta}{n}$.

We can see that while the terms of x with an even order have the same sign, the odd orders are of opposite sign thus the sum of $n_+ + n_-$ leaves only the even orders

$$n_+^{\frac{5}{3}} + n_-^{\frac{5}{3}} \cong \left(\frac{n}{2}\right)^{\frac{5}{3}} \left[2 + \frac{10}{9}\left(\frac{M}{n}\right)^2 + \frac{10}{243}\left(\frac{M}{n}\right)^4\right]$$

So we receive with this expansion

$$\frac{E_K}{V} = \frac{E_0}{V} + \left(\frac{4}{3}\left(3\pi^2\right)^{\frac{2}{3}} \frac{\hbar^2}{2m} n^{-\frac{1}{3}}\right) M^2 + \left(\frac{16}{81}\left(3\pi^2\right)^{\frac{2}{3}} \frac{\hbar^2}{2m} n^{-\frac{7}{3}}\right) M^4$$

For the potential energy we receive

$$\frac{U}{V} = \alpha\left(n_+ n_-\right) = \alpha\left[\left(\frac{n}{2} + \delta\right)\left(\frac{n}{2} - \delta\right)\right] = \alpha\left(\frac{n}{2}\right)^2 - \alpha\delta^2 = \alpha\left(\frac{n}{2}\right)^2 - \alpha\left(\frac{M}{2}\right)^2$$

Finally with swallowing the fourth in alpha we get that the total energy is

$$\frac{E_T}{V} = \frac{E_0}{V} + \alpha n^2 + \left(\frac{4}{3}\left(3\pi^2\right)^{\frac{2}{3}} \frac{\hbar^2}{2m} n^{-\frac{1}{3}} - \alpha\right) M^2 + \left(\frac{16}{81}\left(3\pi^2\right)^{\frac{2}{3}} \frac{\hbar^2}{2m} n^{-\frac{7}{3}}\right) M^4$$

(2) We note that the coefficient of M^4 is always positive but on the other hand for different values of α the coefficient of M^2 can change its sign. The critical value α_c is defined when the coefficient equals zero

$$\alpha_c = \frac{4}{3}\left(3\pi^2\right)^{\frac{2}{3}} \frac{\hbar^2}{2m} n^{-\frac{1}{3}}$$

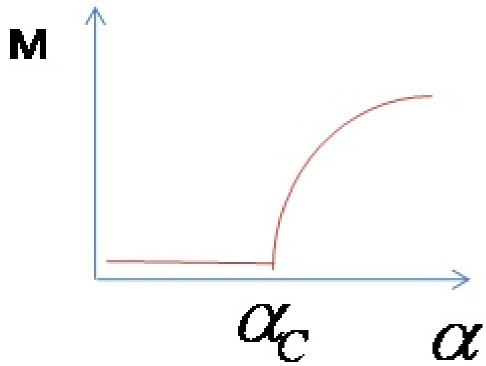
The system can lower its total energy by spontaneously developing magnetization i.e. the coefficient of M^2 needs to be negative in order for that to happen so we need that $\alpha > \alpha_c$.

We can also note that this is a similar expression of the density of states of a 3D Fermi gas with a multiplicity of 2 and so understand that when $\alpha > \frac{1}{N(\epsilon_F)} \equiv \alpha_c$ the system prefers to form in a ferromagnetic form. Finally we define the Stoner criterion as $\alpha = \frac{1}{N(\epsilon_F)}$

(3) Let us denote C_4 as the coefficient of M^4 and $C_2 = \alpha_c - \alpha$ as the coefficient for M^2. The derivative of the energy allows us to see the behavior of the magnetization at minimum energy:

$$\frac{\partial E}{\partial M} = 2C_2 M + 4C_4 M^3 = 0 \Rightarrow M = \sqrt{\frac{-C_2}{2C_4}} = \sqrt{\frac{\alpha - \alpha_c}{2C_4}}$$

and so we get that $\alpha > \alpha_c$.



(4) At finite low temperatures the Fermi-Dirac population function can no longer be described as a step function, we thus use Sommerfeld's approximation

$$\frac{F}{N} = \frac{3}{5}\epsilon_F \left[1 - \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + O\left(\frac{kT}{\epsilon_F}\right)^4\right]$$

Where the small term is $(\frac{kT}{\epsilon_F}) << 1$.

With $\frac{N}{\epsilon_f} = 2m\left(\frac{4\pi}{3h^2}\right)^{\frac{2}{3}} n^{\frac{1}{3}} V$ and $F = E - \frac{\pi^2}{4}\frac{(kT)^2}{\epsilon_F}N$ we write the complete free energy

$$F = F_+ + F_- = E_{0+} + E_{0-} - \frac{\pi^2}{4}(kT)^2(2m)\left(\frac{4\pi}{3h^2}\right)^{\frac{2}{3}} V(n_+^{\frac{1}{3}} + n_-^{\frac{1}{3}})$$

Developing to second order in M we see that: $(n_\pm)^{\frac{1}{3}} \cong \left(\frac{n}{2}\right)^{\frac{1}{3}} \pm \frac{1}{3}\left(\frac{2}{n}\right)^{\frac{2}{3}} M - \frac{2^{\frac{5}{3}}}{9}\frac{M^2}{n^{\frac{5}{3}}}$ and again it is easy to see that the odd terms cancel out. Using previous developments and gathering all the constants together we can notice that when $T \neq 0$ the energy of the system can be written as

$$F = E_0(T) + (\alpha_c(0) - \alpha + \Delta\alpha(kT)^2)M^2$$

And now we can get the condition for a phase transition

$$\alpha_c(0) - \alpha + \Delta\alpha(kT)^2 = 0$$

So we can now denote $\alpha_c(T) = \alpha_c(0) + \Delta\alpha(kT)^2$.