E5963: Stoner ferromagnetism

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The problem:

Consider Fermi gas of N spin $\frac{1}{2}$ electrons at temperature $T = 0$, with N_+ "up" electrons and N_- "down" electrons, such that $N = N_+ + N_-$. Due to the antisymmetry of the total wave function the energy of the system is:

$$U = \alpha \frac{N_+ N_-}{V}$$

Where V is the volume. Note that this interaction favors parallel spin states. Define the magnetization as $M = \frac{N_+ - N_-}{V}$.

(a) Write the total energy $E(M)$ including both the kinetic energy and the interaction, and expand up to 4th order in M.

(b) Find the critical value α_c such that for $\alpha > \alpha_c$ the electron gas can lower its total energy by spontaneously developing magnetization. This is known as the Stoner instability.

(c) Explain the instability qualitatively, and sketch the behavior of the spontaneous magnetization as a function of α.

(d) Repeat (a) at finite but low temperatures T, and find $\alpha_c(T)$ to second order in T.

The solution:

(a) The total energy is the sum of the kinetic and potential energy $E_T = E_K + U$. We will first calculate the Kinetic energy using prior knowledge of fermion gas, then we will add the potential energy. First we note that at ground level the Fermi level is completely populated and thus we can calculate for one of the spin directions:

$$N_\pm = V \int_0^{k_F} \frac{d^3k}{(2\pi)^3} \rightarrow k_F = \left(6\pi^2 n_\pm\right)^{1/3}$$

Where we have used: $\frac{d^3k}{(2\pi)^3} = 4\pi k^2 dk$, we will use this to evaluate the kinetic energy:

$$E_K = 2V \int_0^{k_F} \epsilon(k) \frac{d^3k}{(2\pi)^3} \rightarrow 2V \int_0^{k_F} \frac{\hbar^2 k^2}{2m} 4\pi k^2 dk \propto k_F^5$$

So for $T = 0$ the kinetic energy is:

$$\frac{E_0}{V} = \frac{4\pi\hbar^2}{5m} \left(6\pi^2\right)^{5/3} n^{5/3}$$

Where we used: $\epsilon(k) = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$. We wish to evaluate $n_+^{5/3}$ and $n_-^{5/3}$ at equilibrium: $n_+ = \frac{n}{2} + \delta$, $n_- = \frac{n}{2} - \delta$ so that the magnetization is $M = n_+ - n_- = 2\delta$. Let us expand $n_+^{5/3}$ to 4th order:

$$n_+^{5/3} = \frac{n}{2}^{\frac{5}{3}} \left(1 + \frac{2\delta}{n}\right)^{\frac{5}{3}} \approx \frac{n}{2}^{\frac{5}{3}} \left[1 + \frac{5}{3}x + \frac{\frac{5}{3}\frac{2}{3}}{2}x^2 + \frac{\frac{5}{3}\frac{2}{3}\frac{-1}{3}}{6}x^3 + \frac{\frac{5}{3}\frac{2}{3}\frac{-1}{3}\frac{-4}{3}}{24}x^4\right]$$

$$n_+^{5/3} = \frac{n}{2}^{\frac{5}{3}} \left[1 + \frac{5}{3}x + \frac{5}{9}x^2 - \frac{5}{81}x^3 + \frac{5}{243}x^4 + o(x^5)\right]$$

In a similar fashion we expand $n_-^{5/3}$

$$n_-^{5/3} = \frac{n}{2}^{\frac{5}{3}} \left(1 - \frac{2\delta}{n}\right)^{\frac{5}{3}} \approx \frac{n}{2}^{\frac{5}{3}} \left[1 - \frac{5}{3}x + \frac{\frac{5}{3}\frac{2}{3}}{2}x^2 - \frac{\frac{5}{3}\frac{2}{3}\frac{-1}{3}}{6}x^3 + \frac{\frac{5}{3}\frac{2}{3}\frac{-1}{3}\frac{-4}{3}}{24}x^4\right]$$

$$n_-^{5/3} = \frac{n}{2}^{\frac{5}{3}} \left[1 - \frac{5}{3}x + \frac{5}{9}x^2 + \frac{5}{81}x^3 + \frac{5}{243}x^4 - o(x^5)\right]$$

Where we assigned $x = \frac{2\delta}{n} = \frac{M}{n} << 1$, we can see that while the even orders of x have the same sign, the odd orders are of opposite sign thus when adding we are left with the even orders:

$$n_+^{\frac{5}{3}} + n_-^{\frac{5}{3}} = \frac{n}{2}^{\frac{5}{3}} \left[2 + \frac{10}{9}\left(\frac{M}{n}\right)^2 + \frac{10}{243}\left(\frac{M}{n}\right)^4\right]$$

$$\frac{E_K}{V} = 2\frac{E_0}{V} + \left(\frac{1}{3}\left(3\pi^2\right)^{\frac{2}{3}} \frac{\hbar^2}{2m} n^{-\frac{1}{3}}\right) M^2 + \left(\frac{1}{81}\left(3\pi^2\right)^{\frac{2}{3}} \frac{\hbar^2}{2m} n^{-\frac{7}{3}}\right) M^4$$

the potential energy is:

$$U = \alpha\left(n_+ n_-\right) = \alpha\left[\left(\frac{n}{2} + \delta\right)\left(\frac{n}{2} - \delta\right)\right] = \alpha\left(\frac{n}{2}\right)^2 - \alpha\delta^2 = \alpha\left(\frac{n}{2}\right)^2 - \alpha\left(\frac{M}{2}\right)^2$$

The total energy is:

$$E_T = 2\frac{E_0}{V} + \alpha\frac{n}{2}^2 + \left(\frac{1}{3}\left(3\pi^2\right)^{\frac{2}{3}} \frac{\hbar^2}{2m} n^{-\frac{1}{3}} - \frac{\alpha}{4}\right) M^2 + \left(\frac{1}{81}\left(3\pi^2\right)^{\frac{2}{3}} \frac{\hbar^2}{2m} n^{-\frac{7}{3}}\right) M^4$$

(b) We can note that the coefficient of M^4 is always positive, but for different values of α the coefficient of M^2 can change it's sign. The critical value α_c is defined when the coefficient equals zero.

$$\alpha_c = \frac{4}{3}\left(3\pi^2\right)^{\frac{2}{3}} \frac{\hbar^2}{2m} n^{-\frac{1}{3}}$$

We can note that this is a similar expression of the density of states of a 3D Fermi gas with a multiplicity of 2:

$$\frac{1}{\mathcal{N}} = \frac{4}{3}\left(3\pi^2\right)^{\frac{2}{3}} \frac{\hbar^2}{2m} n^{-\frac{1}{3}}$$

We define the Stoner criteria as $\alpha > \frac{1}{\mathcal{N}_{\epsilon F}}$.

(c) Let us denote C_4 as the coefficient of M^4 and $C_2 = \alpha_c - \alpha$ as the coefficient for M^2. the derivative of the energy allows us to see the behavior of the magnetization at minimum energy:

$$\frac{\partial E}{\partial M} = 2C_2 M + 4C_4 M^3 = 0 \rightarrow M^2 = \frac{-C_2}{2C_4} = \frac{\alpha - \alpha_c}{2C_4}$$

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Figure 1: Change of magnetization over a change of alpha

(d) At finite low temperatures the Fermi-Dirac population function can no longer be described as a step function, we thus use Zommerfeld's approximation:

$$\frac{F}{N} = \frac{3}{5}\epsilon_F \left[1 - \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + O\left(\frac{kt}{\epsilon_F}\right)^4\right]$$

Where the small term is: $\left(\frac{kt}{\epsilon_F}\right) << 1$. By implementing Zommerfeld's approximation and using $\frac{N}{\epsilon_F} = 2m\left(\frac{4\pi}{3h^2}\right)^{\frac{2}{3}} n^{\frac{1}{3}} V$
The complete free energy is:

$$F = F_+ + F_- = E_{0+} + E_{0-} - \frac{\pi^2}{4}(kT)^2\left(\frac{4\pi}{3h^2}\right) V \left(n_+^{\frac{1}{3}} + n_-^{\frac{1}{3}}\right)$$

Developing to second order in M we see that: $n_\pm \approx \left(\frac{n}{2}\right)^{\frac{1}{3}} - \frac{2^{\frac{2}{3}}}{9}\frac{M^2}{n^{\frac{5}{3}}}$. The energy of the system can now be written as:

$$E = E_0(T) + (\alpha_c - \alpha + \gamma(kT)^2)\left(\frac{M}{2}\right)^2 + C_4 M^4$$

Where: $E_0(T)$ is the constant energy in the system: $E_0(T) = 2\frac{E_0}{V} + \alpha\frac{n}{2}^2 - \frac{\pi^2}{4}(kT)^2\left(\frac{4\pi}{3h^2}\right) V \left(\frac{n}{2}\right)^{\frac{1}{3}}$ γ is the correction due to nonzero temperatures: $\gamma = \frac{2^{\frac{2}{3}}}{9}\frac{\pi^2}{4n^{\frac{5}{3}}}\left(\frac{4\pi}{3h^2}\right) V \left(\frac{n}{2}\right)^{\frac{1}{3}}$
A phase transition will occur when:

$$\alpha_c - \alpha + \gamma(kT)^2 = 0$$

So, we can now assign $\alpha_c(T) = \alpha_c(0) + \gamma(kT)^2$. from this expression we see that the higher the temperature is, we will need a stronger coupling constant.