E5966: Long Range Interaction Ising Model 1D, Domain Walls

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The problem:

Consider the one dimensional Ising model with the hamiltonian

$$H = -\sum_{n,n'} J(n - n')\sigma(n)\sigma(n')$$

with $\sigma(n) = \pm 1$ at each site n and long range interaction $J(n) = b/n^\gamma$ with $b > 0$. Find the energy of a domain wall at $n = 0$, i.e. all the $n < 0$ spins are down and the others are up. Show that the standard argument for the absence of spontaneous magnetization at finite temperature fails if $\gamma < 2$.

The solution:

The standard argument for the absence of spontaneous magnetization is that in $1D$, it is always favourable (for any finite Temperature) to create a domain wall at some location (lets call it $n = 0$) thus gaining entropy at the expense of a small energy increase.

$F = E - TS$ is the free energy we need to minimize. Since the energy gain (in comparison to the ground state) in creating a domain wall is $2J$ (in the standard argument) and the entropy becomes $\ln(N)$ where N is the number of spins and also the number of positions the wall can assume. the change in free energy is $\Delta F = 2J - T\ln(N)$ and since N is huge, this is negative for any finite temperature, making this transition favorable.

In our long range interaction model, the ground state energy is the summing of all interactions in any pair of spins, all being the same sign. We will count over all N spins, making sure we dont double count by demanding $n' > n$ in our sum:

$$E_0 = -\sum_{n=-\infty}^{\infty} \sum_{n'=n+1}^{\infty} \frac{b}{(n' - n)^\gamma} \tag{1}$$

We will not solve this sum.

When creating a domain wall at $n = 0$ we flip the sign of the interaction between every pair on opposite sides of the wall, thus paying a double price (-1 goes to 1, the difference is 2). We sum over all N spins (we assume N spins on every side, but obviously it doesnt have to be and it doesnt matter):

$$E_1 = E_0 + 2\sum_{n=0}^{N} \sum_{n'=1}^{N} \frac{b}{(n' + n)^\gamma} \tag{2}$$

In order to go to an integral form, we make n continuous by going to new variables representing spin location. We introduce spin separation a and define $x = na$ and $y = n'a$. $dx = dy = a$.

$$E_1 = E_0 + 2ba^{\gamma-2} \int_0^{Na} dx \int_a^{Na} dy \frac{1}{(x+y)^\gamma} \tag{3}$$

This integral is solved differently for different γ.

For $\gamma > 2$:

$$\Delta E = 2\frac{b}{(\gamma-1)(\gamma-2)} \tag{4}$$

For $\gamma = 2$:

$$\Delta E = 2b\ln(N) \tag{5}$$

Under the value of $\gamma = 2$ the divergence in N is even stronger. For example the $1 < \gamma < 2$ case gives:

$$\Delta E = 2N^{2-\gamma}\frac{b}{(\gamma-1)(\gamma-2)} \tag{6}$$

for the cases of $\gamma \leq 2$ we see that it is not favorable, in terms of free energy, to create a domain wall:

$$\Delta F = -T\ln(N) + 2b\ln(N) \tag{7}$$

$$\Delta F = -T\ln(N) + 2bN^{2-\gamma}/(\gamma-1)(\gamma-2) \tag{8}$$

in the first case it depends on the value of T. In the second case it is never (as N is huge) favorable to create domain walls, thus breaking the argument.