E5966:Ising model on bipartite lattice, domain walls

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The problem:

(a) Consider the Ising spin model on a bipartite lattice, i.e. it has two sublattices A,B such that each site on lattice A has its nearest neighbors on sublattice B, and vice versa. The Hamiltonian is

$$\mathcal{H} = -J \sum_{\mathbf{n},\boldsymbol{\delta}} \sigma(\mathbf{n})\sigma(\mathbf{n}+\boldsymbol{\delta}) - h \sum_{\mathbf{n}} \sigma(\mathbf{n}) \tag{1}$$

where $\mathbf{n}$ are the lattice sites, $\boldsymbol{\delta}$ labels the nearest neighbors, h is proportional to a magnetic field and $\sigma(\mathbf{n}) = \pm 1$. For $h = 0$ show that the free energy satisfies $F(J,T) = F(-J,T)$, hence the critical temperatures satisfy $T_c^F = T_c^{AF}$ for the ferromagnetic ($J > 0$) and anti-ferromagnetic ($J < 0$) transitions. Define the order parameters at $T < T_c$ and the magnetic susceptibilities to h at $T > T_c$ and find their relationship, if any.

(b) Consider the one dimensional Ising model with the Hamiltonian $\mathcal{H} = -\sum_{n,n'} J(n-n')\sigma(n)\sigma(n')$ with $\sigma(n) = \pm 1$ at each site n and $J(n) = b/n^{\gamma}$ is a long range interaction and $b > 0$. Find the energy of a domain wall (i.e. $n < 0$ spins are $-$ and $n \geq 0$ are $+$) and show that the argument for the absence of spontaneous magnetization at finite temperatures fails when $\gamma < 2$.

The solution:

(a) Ferromag $\rightarrow AF$:

$$Z = \sum_{\sigma_n = \pm 1} e^{\beta J \sum_n \sigma_n \sigma_{n+1}} \underbrace{\longrightarrow}_{\sigma_n \rightarrow -\sigma_n} \sum_{\sigma_n = \pm 1} e^{-\beta J \sum_n \sigma_n \sigma_{n+1}} \tag{2}$$

.

so that bi-partite lattice give:

$$F(J) = F(-J) \tag{3}$$

(b)

$$\mathcal{H} = -\sum_{n,n'} \frac{b}{|n-n'|}(n)\sigma(n') \tag{4}$$

where $\sigma(n) = \pm 1$

$$E_{wall} = \underbrace{\downarrow\downarrow\downarrow \ldots}_{\frac{N}{2}} | \underbrace{\uparrow\uparrow\uparrow \ldots}_{\frac{N}{2}}$$

where $\frac{N}{2}$ spins is up and $\frac{N}{2}$ is down

$$E_{wall} = 2b \sum_{n=-\infty}^{-1} \sum_{n'=0}^{\infty} \frac{1}{|n-n'|^{\gamma}} \backsimeq 2b \int_{-\infty}^{0} dn \int_{0}^{\infty} dn' \frac{1}{|n-n'|^{\gamma}} \tag{5}$$

The convergence at $n \to \infty$ needed, hence, $\sum_n \to \int dn$ is valid
so we got:

$$E_{wall} = 2b \int_{-\infty}^{0} dn \frac{-1}{\gamma-1} \frac{1}{(n'-n)^{\gamma-1}} \Bigg|_{n'=0}^{\infty} \tag{6}$$

(i) If $\gamma > 1$:

$$E_{wall} = \frac{2b}{\gamma-1} \int_{-\infty}^{0} dn \frac{1}{-n}^{\gamma-1} = \frac{2b}{\gamma-1} \int_{0}^{\infty} dn \frac{1}{n^{\gamma-1}} = \frac{2b}{(\gamma-1)(\gamma-2)\frac{-1}{n^{\gamma-2}}} \Bigg|_{n'=0}^{\infty} \tag{7}$$

(ii) If $\gamma < 2$:
Where wall entropy $\backsim ln(N)$ now :

$$F_{wall} \backsim N^{2-\gamma} - K_B ln(N) > 0 \tag{8}$$

no walls are created when there are long-range intraction!