Ex5811: Mechanical model for symmetry breaking

Submitted by: Neta Shaul

The problem

An airtight piston of mass M is free to move inside a cylindrical tube of cross sectional area a. The tube is bent into a semicircular shape of radius R. On each side of the piston there is an ideal gas of N atoms at a temperature T. The angular position of the piston is φ (see figure). The gravitation field of Earth exerts a force Mg on the piston, while its effect on the gas particles can be neglected. The partition function of the system can be written as dφ integral over exp[-A(φ)]. The variable φ is regarded as the "order parameter" of the system. A small difference ΔN in the occupation of the two sides is regarded as the conjugate field. The susceptibility is defined via the relation $\langle\varphi\rangle \approx \chi\Delta N$.

1) Write an explicit expression for A(φ).
2) Find the coefficients in the expansion : $A(\varphi) = (a/2)\varphi^2 + (u/4)\varphi^4 - h\varphi$.
3) Deduce what is the critical temperature T_c.
4) Using Gaussian approximation find what is χ for T$>T_c$.
5) Using Gaussian approximation find what is χ for T$<T_c$.
6) Sketch a plot of χ versus T indicating by dashed lines the Gaussian approximations and by solid line the expected exact result. Write what is the range ΔT around T_c where the Gaussian approximation fails.
7) What is the way to take the "thermodynamic limit" such as to have a phase transition at finite temperature ?
8) In reality, as the temperature is lowered, droplets condense on the walls of the left (larger) chamber. What do you expect to find in the right chamber (gas? liquid? both?).

Guidlines: In items (4) and (5) simplify the result assuming T˜T_c and express it in terms of T_c and T-T_c. The final answer should include one term only. Care about numerical prefactors - their correctness indicates that the algebra is done properly. In item (7) you are requested to identify the parameter that shouls be taken to infinity in order to get a "phase transition". Please specify what are the other parameters that should be kept constant while taking this limit.

The solution

Section 1

As noted above, the patrition function can be written as an integral over φ:

$$Z = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-A(\varphi)} d\varphi \tag{1}$$

thus to get $A(\varphi)$, we shall find the partition function for a constant angle Z_φ.

For a constant φ the there is no interaction between the left and right side of the tube so the contributions to Z_φ can be diveded to three:

$$Z_\varphi = Z_l \cdot e^{-\beta MgRcos(\varphi)} \cdot Z_r \tag{2}$$

where $Z_{l/r}$ are the partition functions of the gas in the tube for the left and right side respectively.

The gas in the tube is ideal so:

$$Z_{l/r} = \frac{1}{N_{l/r}!}\left(\frac{aR(\frac{\pi}{2} \pm \varphi)}{\lambda_T^3}\right)^{N_{l/r}} \tag{3}$$

we are now ready to write the expression for $A(\varphi)$:

$$\boxed{A(\varphi) = -ln(Z_\varphi) = ln(N_l!) - N_l ln\left(\frac{aR(\frac{\pi}{2}+\varphi)}{\lambda_T^3}\right) + \beta MgRcos(\varphi) + ln(N_r!) - N_r ln\left(\frac{aR(\frac{\pi}{2}-\varphi)}{\lambda_T^3}\right)} \tag{4}$$

Section 2

We shall now find an approximation to $A(\varphi)$ in form suggested above, to make the process as clear as possible we will do it in three steps.

Let us first note that through out this exercise we are intersted in $\langle\varphi\rangle$, since expressions that are constant in φ will factorize in the partion function we can omit them:

$$A(\varphi) = -N_l ln\left(1 + \frac{2\varphi}{\pi}\right) + \beta MgRcos(\varphi) - N_r ln\left(1 - \frac{2\varphi}{\pi}\right) \tag{5}$$

In addtion, we note that it is given that at each side there are N particles but since $\langle\varphi\rangle \approx \chi\Delta N$, we assume a small change in occupation while keeping the total number of particle const in order to find χ:

$$N_l = N + \frac{\Delta N}{2} \text{ and } N_r = N - \frac{\Delta N}{2} \tag{6}$$

inserting back to $A(\varphi)$ and reordering:

$$A(\varphi) = \beta MgRcos(\varphi) - Nln\left[1 - \left(\frac{2\varphi}{\pi}\right)^2\right] - \Delta N\left[ln\left(1 + \frac{2\varphi}{\pi}\right) - ln\left(1 - \frac{2\varphi}{\pi}\right)\right] \tag{7}$$

For the last step we use the taylor expansion, it is important to note that since ΔN takes the role of the external field it is coupled only to the linear term of φ thus for the right most term in eq (7) only the first order is taken while for the others we keep up to the fourth order:

$$\boxed{A(\varphi) \approx \frac{1}{2}\left(\frac{8N}{\pi^2} - \frac{MgR}{T}\right)\varphi^2 + \frac{1}{4}\left(\frac{MgR}{6T} + \frac{32N}{\pi^4}\right)\varphi^4 - \left(\frac{2\Delta N}{\pi}\right)\varphi} \tag{8}$$

Section 3

For the critical temperture we have $a(T_c) = 0$, so by observing eq (8):

$$\boxed{T_c = \frac{\pi^2 MgR}{8N}} \tag{9}$$

Section 4

The Gaussian approximation assume that the main contribution to intgral in eq (1) comes from the minimum of $A(\varphi)$ and so we shall find it. Requiring that $A'(\varphi) = 0$ gives:

$$a\varphi + u\varphi^3 - h = 0 \tag{10}$$

for $T > T_c$ we have $a > 0$ thus the only solution for:

$$a\varphi + u\varphi^3 = 0 \tag{11}$$

is $\varphi = 0$ which implies that $a\varphi + u\varphi^3$ has no local minimum/maximum, so we conclude eq (10) has only one solution. Since $A(\varphi)$ has only one miminum we ignore the fourth order and eq (10) becomes:

$$A'(\varphi) = a\varphi - h = 0 \tag{12}$$

which leads to:

$$\bar{\varphi} = \frac{h}{a} \tag{13}$$

Thus according to gaussain approximation:

$$Z = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-A(\varphi)} d\varphi \approx \int_{-\infty}^{\infty} e^{-\frac{a}{2}(\varphi-\bar{\varphi})^2 + h(\varphi-\bar{\varphi})} d\varphi = \sqrt{\frac{2\pi}{a}} e^{\frac{h^2}{2a}} \tag{14}$$

at last we have are ready to calculate $\langle\varphi\rangle$:

$$\langle\varphi\rangle = \frac{\partial ln(Z)}{\partial h} = \frac{h}{a} = \frac{2}{\pi a}\Delta N \tag{15}$$

Following the guidelines, we assume $T \sim T_c$ and approximate a by:

$$a \approx \frac{8N}{\pi^2} \cdot \frac{T - T_c}{T_c} \tag{16}$$

we conclude:

$$\boxed{\chi = \frac{2}{\pi a} \propto \frac{1}{N} \cdot \frac{T_c}{T - T_c}} \tag{17}$$

Section 5

We shall follow the same steps as in section 4, except this time $T < T_c$ and thus $a < 0$ so for h small enough, eq (10) has 3 three solutions and as opposed to the previous case we cannot ignore the fourth order.

To solve eq (10) we treat h as a small perturbation and solve only up to leading order in h (that is the zero order) which gives minimum points in:

$$\bar{\varphi}_{\pm} = \pm\sqrt{\frac{|a|}{u}} \tag{18}$$

According to the gaussian approximation each miminum contribute a gaussian integral, since this time the partition function becomes a sum of integral it is important to keep zero orders as well:

$$A_{\pm}(\varphi) \approx |a|(\varphi - \bar{\varphi}_{\pm})^2 - h(\varphi - \bar{\varphi}_{\pm}) - h\bar{\varphi}_{\pm} - \frac{|a|^2}{4u} \tag{19}$$

so the partition function we get:

$$Z = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-A(\varphi)} d\varphi \approx \int_{-\infty}^{\infty} e^{-A_-(\varphi)} + \int_{-\infty}^{\infty} e^{-A_+(\varphi)} = 2\sqrt{\frac{\pi}{a}} e^{\frac{|a|^2}{4u}} \cdot e^{\frac{h^2}{4|a|}} cosh\left(h\sqrt{\frac{|a|}{u}}\right) \tag{20}$$

and as before $\langle\varphi\rangle$ is given by:

$$\langle\varphi\rangle = \frac{\partial ln(Z)}{\partial h} = \frac{h}{2a} + \sqrt{\frac{|a|}{u}} tanh\left(h\sqrt{\frac{|a|}{u}}\right) \approx \left(\frac{1}{\pi|a|} + \frac{2|a|}{\pi u}\right)\Delta N \tag{21}$$

Following the guidelines, we assume $T \sim T_c$ and approximate u by:

$$u \approx \frac{4N}{3\pi^4}(24+\pi^2) \tag{22}$$

we conclude:

$$\boxed{\chi \approx \frac{\pi}{8N} \cdot \frac{T_c}{T-T_c} + \frac{12\pi}{24+\pi^2} \cdot \frac{T-T_c}{T_c}} \tag{23}$$

Section 6

The condition for the gaussian approximation is that around $\bar{\varphi}/\bar{\varphi}_\pm$ the quartic term in $A(\varphi)$ its neglible. We are interested in the region $T \sim T_c$ and we have two cases to check.
Let us first examine the case $T > T_c$, we have $\bar{\varphi} \approx 0$ and the condition becomes:

$$|a|\varphi^2 >> |u|\varphi^4 \tag{24}$$

We interpret "around $\bar{\varphi}$" by $|\varphi - \bar{\varphi}| < \sigma$ where σ is the dispersion of φ.
By eq (14) $\sigma^2 = \frac{1}{a} \propto \frac{1}{n} \cdot \frac{T_c}{T-T_c}$ and $\left|\frac{a}{u}\right| \propto \frac{T-T_c}{T_c}$, inserting back leads to:

$$\left|\frac{T-T_c}{T_c}\right| >> \frac{1}{\sqrt{N}} \tag{25}$$

For the case $T < T_c$ we have $\bar{\varphi}_\pm \approx \pm\sqrt{\left|\frac{a}{u}\right|}$, writing $A(\varphi)$ around $\bar{\varphi}_\pm$ gives:

$$A_{h=0}(\bar{\varphi}_\pm + \varphi) = |a|\varphi^2 + \frac{u}{4}\varphi^4 \pm \sqrt{|ua|}\varphi^3 + const \tag{26}$$

so the condition becomes:

$$|a|\varphi^2 >> \left|\frac{u}{4}\varphi^4 \pm \sqrt{|ua|}\varphi^3\right| \tag{27}$$

By inspecting eq (19) we see $\sigma^2 = \frac{1}{2|a|} \propto \frac{1}{N} \cdot \left|\frac{T_c}{T-T_c}\right|$, inserting back to the condition gives the same result as for $T > T_c$ which leads us to conclude that the gaussian approximation fails in the region:

$$\boxed{\left|\frac{T-T_c}{T_c}\right| << \frac{1}{\sqrt{N}}} \tag{28}$$

Following the guidelines we approximate $\chi(T < T_c)$ to one term, the above discussion lead us to assume inequality (25) thus:

$$\chi(T < T_c) \approx \frac{12\pi}{24+\pi^2} \cdot \frac{T-T_c}{T_c} \tag{29}$$

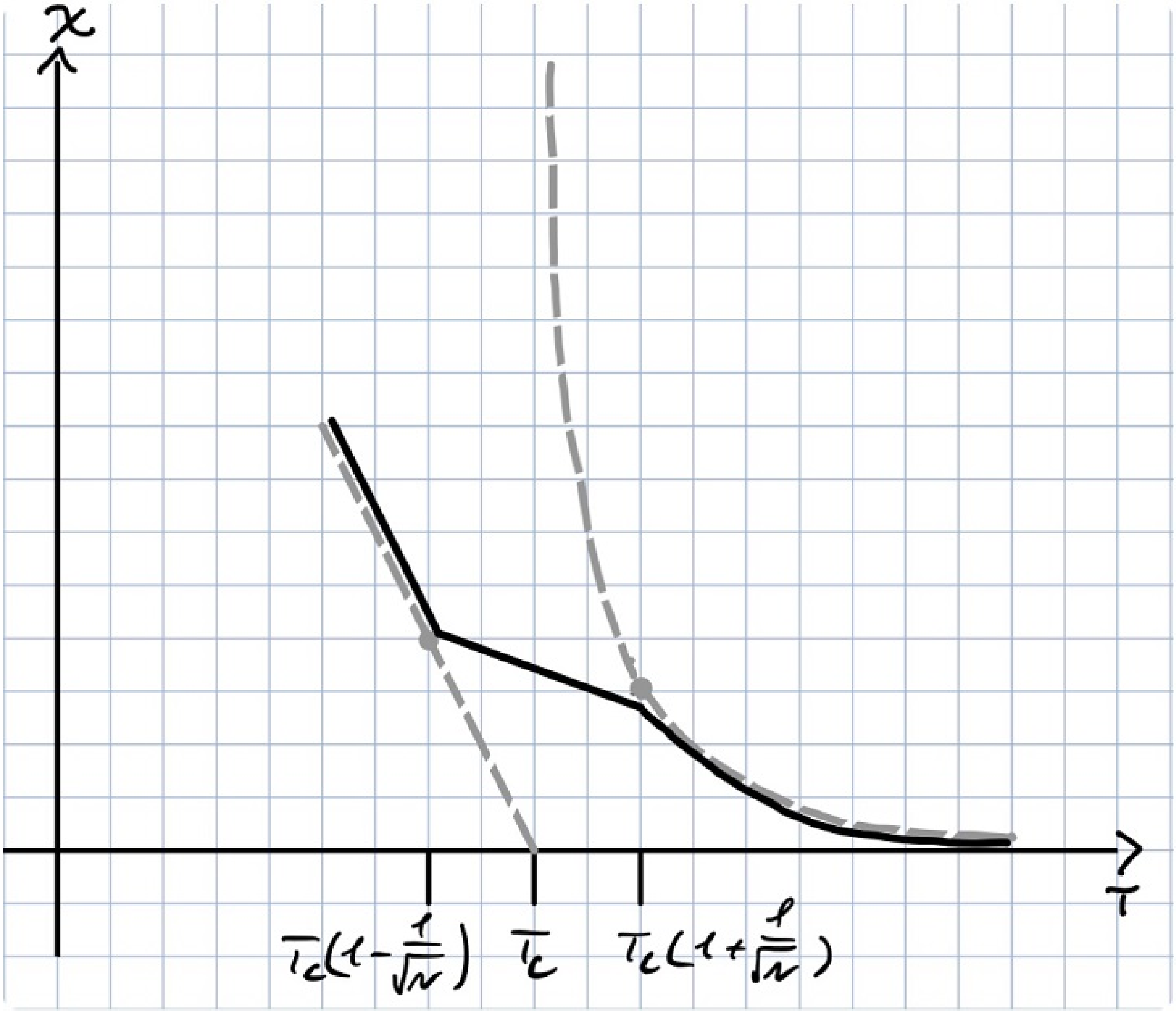

Figure 1: Dashed lines for the gaussian approximation and solid line for the expected graph

Last we shall explain the requested graph that is presented in Figure 1, the dashed lines indicate the result we got from the gaussian approximation.
We can see that T_c is a discontinuous point which disagree with our expectation for $\chi(T)$ to be a continuous function. We explain this result by provoking the previous discussion of the validity of the gaussian approximation where we have seen that it fails around T_c. Out side the region where the approximation fails, inequality (25), we assume that $\chi(T)$ acts according to the result, inside that region we drew an interpolation thus our expected result is a line that follow the approximation out side the region and a line that connects the enpoints, we note that the slope of the line is independet of N.

Section 7

Taking the "Thermodynamic limit" means we take the the limit $N \to \infty$ while keeping the density $n = \dfrac{N}{\pi R a}$ constant. To get a phase transition at a finite temperture we have to keep T_c constant and by eq (9):

$$T_c \propto \frac{R}{N} \tag{30}$$

so indeed by keep the density constant we get a finite T_c even at the limit $N \to \infty$, more specifically we keep:

$$\frac{N}{R} = const \tag{31}$$

It is nice to see that indeed we have a phase transition at this limit by inspecting:

$$\lim_{N\to\infty} \chi = \begin{cases} 0 & T > T_c \\ \frac{12\pi}{24+\pi^2} \cdot \frac{T-T_c}{T_c} & T < T_c \end{cases} \tag{32}$$

Section 8

Assuming the system is in equiliberium we have $T_l = T_r$ and $P_r = P_l + \dfrac{Mg|sin(\varphi)|}{a}$, here a denots the cross section of the tube. Remmeber that the state of coexistence of liquid and gas is a line $P(T)$, assuming the state of the side with the larger volume is on the line, since $P_r > P_l$ the state of the side with the smaller volume must be above the line thus contains liquid only.