Ex5811: Mechanical model for symmetry breaking

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The problem:

An airtight piston of mass M is free to move inside a cylindrical tube of cross sectional area a. The tube is bent into a semicircular shape of radius R. On each side of the piston there is an ideal gas of N atoms at a temperature T. The angular position of the piston is φ (see figure). The gravitation field of Earth exerts a force Mg on the piston, while its effect on the gas particles can be neglected. The partition function of the system can be written as dφ integral over exp[-A(φ)]. The variable φ is regarded as the "order parameter" of the system. A small difference ΔN in the occupation of the two sides is regarded as the conjugate field. The susceptibility is defined via the relation $\langle\varphi\rangle \approx \chi\Delta N$.

(1) Write an explicit expression for A(ϕ).
(2) Find the coefficients in the expansion : $A(\phi) = (a/2)\phi^2 + (u/4)\phi^4 - h\phi$.
(3) Deduce what is the critical temperature T_c.
(4) Using Gaussian approximation find what is χ for T>T_c.
(5) Using Gaussian approximation find what is χ for T<T_c.
(6) Sketch a plot of χ versus T indicating by dashed lines the Gaussian approximations and by solid line the expected exact result. Write what is the range ΔT around T_c where the Gaussian approximation fails.
(7) What is the way to take the "thermodynamic limit" such as to have a phase transition at finite temperature ?
(8) In reality, as the temperature is lowered, droplets condense on the walls of the left (larger) chamber. What do you expect to find in the right chamber (gas? liquid? both?).

Guidlines: In items (4) and (5) simplify the result assuming T˜T_c and express it in terms of T_c and T-T_c. The final answer should include one term only. Care about numerical prefactors - their correctness indicates that the algebra is done properly. In item (7) you are requested to identify the parameter that shouls be taken to infinity in order to get a "phase transition". Please specify what are the other parameters that should be kept constant while taking this limit.

The solution:

(1) We know that the free energy of the system equals the sum of the free energy of the two gases and the energy of the piston so we can write :

$$A(\phi) = N_1 ln\left(\frac{N_1\lambda_T^3}{aR(\frac{\pi}{2}-\phi)}\right) - N_1 + N_2 ln\left(\frac{N_2\lambda_T^3}{aR(\frac{\pi}{2}+\phi)}\right) - N_2 + \beta MgRcos(\phi) \tag{1}$$

(2) We define the following :

$$N_1 = N - (1/2)\Delta N \tag{2}$$

$$N_2 = N + (1/2)\Delta N \tag{3}$$

Taylor expansion:

$$cos(\phi) = 1 - \frac{\phi^2}{2} + \frac{\phi^4}{24} + ... \tag{4}$$

$$ln(1+x) = x - x^2 + x^3 - x^4 + .. \tag{5}$$

We insert equations (3),(4),(5) in equation (2) and after some algebra we get the following :

$$A(\phi) = (N-\Delta N)ln\left(\frac{1-\frac{\Delta N}{N}}{1-\frac{2\phi}{\pi}}\right) + (N+\Delta N)ln\left(\frac{1+\frac{\Delta N}{N}}{1+\frac{2\phi}{\pi}}\right) + \beta MgR(-\frac{\phi^2}{2}+\frac{\phi^4}{24}) \tag{6}$$

We omit constants that are dependent on ΔN :

$$A(\phi) = -(N-\Delta N)ln\left(1-\frac{2\phi}{\pi}\right) - (N+\Delta N)ln\left(1+\frac{2\phi}{\pi}\right) + \beta MgR(-\frac{\phi^2}{2}+\frac{\phi^4}{24}) \tag{7}$$

Now we use equation (6) and after little algebra :

$$A(\phi) = (1/2)\left(\frac{8N}{\pi^2} - \beta MgR\right)\phi^2 + (1/4)\left(\frac{32}{\pi^4}+\frac{\beta MgR}{6}\right)\phi^4 - \left(\frac{4\Delta N}{\pi}\right)\phi \tag{8}$$

Its easy to see that :

$$a = \frac{8N}{\pi^2} - \beta MgR = \left(\frac{8N}{\pi^2}\right)\left(\frac{T-T_c}{T}\right) \qquad u = \frac{32N}{\pi^4}+\frac{\beta MgR}{6} \qquad h = \frac{4\Delta N}{\pi} \tag{9}$$

(3) We can calculate T_c be equating the coeffecient "a" to zero :

$$T_c = \frac{MgR\pi^2}{8N} \tag{10}$$

(4) For T>T_c , a>0 and u=0 :

$$Z = \int d\phi e^{-A(\phi)} = \int d\phi e^{-(a/2)\phi^2 - h\phi} \tag{11}$$

The minimum of $A(\phi)$ is when derivative of A is zero, ϕ = -$\frac{h}{a}$.
If there is a difference in the number of molecules then we can write :

$$h = \frac{2}{\pi}\Delta N \tag{12}$$

We use the last equation and get :

$$Z = \int d\phi e^{-(a/2)(\phi+\frac{h}{a})^2 + (a/2)(\frac{h}{a})^2} = \sqrt{\frac{2\pi}{a}}e^{\frac{h^2}{2a}} \tag{13}$$

$$ln(Z) = \frac{h^2}{2a} + (1/2)ln\left(\frac{2\pi}{a}\right) \tag{14}$$

$$\langle\phi\rangle = \frac{dlnZ}{d\Delta N} = \left(\frac{2}{a\pi}\right)\Delta N \tag{15}$$

From the last equation we can conclude that :

$$\chi = \frac{2}{\pi a} = \frac{\pi}{4N}\left(\frac{T_c}{T-T_c}\right) \tag{16}$$

(5) For T<T_c , a<0 and we choose the field h equal zero. There are two gaussians that we have to take in account becuase they contribute the most to the propability :

$$A(\phi) = (a/2)\phi^2 + (u/4)\phi^4 \tag{17}$$

In order to calculate the peak's values, we demand :

$$\frac{dA}{d\phi} = 0 \Rightarrow a\phi - u\phi^3 = 0 \Rightarrow \phi = \pm\sqrt{\frac{|a|}{u}} \tag{18}$$

As we said before the main contribution come from two gaussians that we had calculated there mean values above.

$$A(\phi_{extremum}) = -\frac{|a|^2}{2u} + \frac{u}{4}\frac{|a|^2}{u^2} \mp h\sqrt{\frac{|a|}{u}} = -\frac{|a|^2}{4u} \mp h\sqrt{\frac{|a|}{u}} \tag{19}$$

We use the gaussian approximation and assume that the distance between the two gaussians is much larger than their width (we keep the fourth order in order to get a symmetry brekaing) : we calculate Z :

$$Z = \int d\phi e^{-(a/2)\phi^2-(u/4)\phi^4} = \int d\phi_{right} + \int d\phi_{left} = 2\sqrt{\frac{2\pi}{a}}e^{-\frac{|a|^2}{4u}} cosh\left(h\frac{|a|}{u}\right) \tag{20}$$

$$\langle\phi\rangle = \frac{dlnZ}{d\Delta N} = tanh\left(h\frac{|a|}{u}\right)\sqrt{\frac{|a|}{u}} = \frac{|a|}{u}\frac{2}{\pi}\Delta N = \left(\frac{12\pi}{24+\pi^2}\right)\left(\frac{|T-T_c|}{T_c}\right)\Delta N \tag{21}$$

$$\chi = \left(\frac{12\pi}{24+\pi^2}\right)\left(\frac{|T-T_c|}{T_c}\right) \tag{22}$$

(6) See the Figure below.

(7) In order to have a phase transition at finite temperature we have to approach the thermodynamic limit by taking N to infinity while keeping T_c constant. The point here that at finite N the gaussian approximation is right only at range that is far away from T_c (as we had seen in (6)).
When N starts to increase the gaussian approximation will be better, until we reach the thermodynamic limit at which the approximation will be right for every temprature. At the thermodynamic limit χ will the have values according to equations (16) and (22), zero for T>T_c and a constant for T<T_c .

(8) As the temperature is lowered, droplets will condense also on the walls of the right chamber becuase :

$$P_1 > P_2 \qquad\qquad V_2 > V_1 \tag{23}$$

So we expect to find liquid in the right chamber.

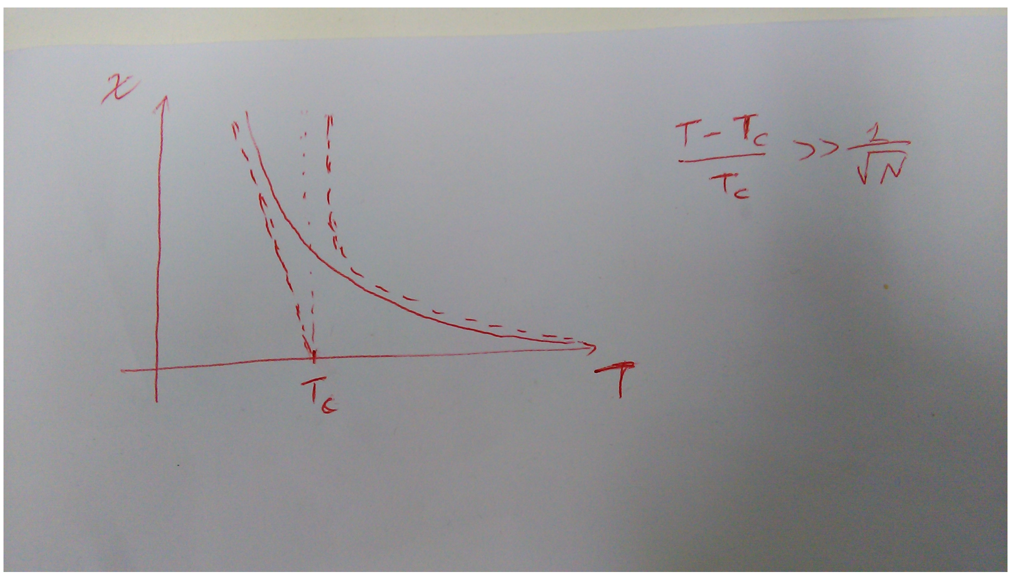

Figure 1: A plot for χ versus T