E5721: Ising antiferromagnet

Submitted by: Alona Petrushansky

The problem:

Consider Ising model on a 2D lattice with antiferromagnetic interaction ($\epsilon = -\epsilon_0$). You can regard the lattice as composed of two sublattices A and B, such that $M = \frac{1}{2}(M_A + M_B)$ is the averaged magnetization per spin, and $M_s = \frac{1}{2}(M_A - M_B)$ is the staggered magnetization

(1) Explain the claim: for zero field ($h = 0$), Ising antiferromagnet is the same as Ising ferromagnet, where M_s is the order parameter.
(2) Given h and ϵ_0, find the coupled mean-field equations for M_A and M_B. Write the expression for $M_s(T)$ for $T \sim T_c$, based on the familiar solution of the ferromagnetic case.

(3) Find the critical temperature T_c for $h = 0$, and also for small h. Hints: for $h = 0$ use the same procedure of expanding $\text{arctanh}(x)$ as in the ferromagnetic case; for small h you may use the most extreme simplification that does not give a trivial solution.

(4) Find the critical magnetic field h_c above which the system no longer acts as an antiferromagnet at zero temperature.

(5) Find an expression for the susceptibility $\chi(T)$, expressed as a function of the staggered magnetization $M_s(T)$.

(6) In the region of $T \sim T_c$ give a linear approximation for $1/\chi$ as a function of the temperature T

The solution:

(1) The Hamiltonian is:

$$\mathcal{H} = -\epsilon \sum s_i s_j - h \sum_i s_i \tag{1}$$

For zero magnetic field and low temperature the arrangement of spins depends on the interaction term $\epsilon s_i s_j$ of the spins. By regard the lattice as composed of two sublattices A and B, where each spin A have neighbors spins B we have:
for ferromagnet $\epsilon > 0$, we have minimum energy when $\langle s_A \rangle s_B > 0$, as for antiferromagnet, $\epsilon < 0$ and therefore minimum energy will obtain when$\langle s_A \rangle s_B < 0$, from this claim we can see that

$$\mathcal{H}^{AF}(h=0) = \mathcal{H}^{F}(h=0) \tag{2}$$

(2) The Hamiltonian for sub lattice A is of the form:

$$\mathcal{H} = -\epsilon c \langle \sigma_B \rangle \sum_i \sigma_{A_i} - h \sum_i \sigma_{A_i} = -\sum_i \sigma_{A_i} (\underbrace{\epsilon c \langle \sigma_B \rangle + h}_{\bar{h}}) = -\sum_i \sigma_{A_i} \bar{h} \tag{3}$$

the Hamiltonian of one spin from sub lattice A is of form:

$$\mathcal{H}_{\mathrm{i}} = -\epsilon c \left\langle \sigma_B \right\rangle \sigma_{A_i} - h\sigma_{A_i} = -\sigma_{A_i}(\underbrace{\epsilon c \left\langle \sigma_B \right\rangle + h}_{\overline{h}}) = -\sigma_{A_i}\overline{h} \tag{4}$$

for summation of index i over spin up(+) and down(-) we have

$$Z = \sum_i e^{-\beta H_i} = 2cosh(\beta\overline{h}) \tag{5}$$

$$M_{A/B} = \left\langle \sigma_{A/B} \right\rangle = \frac{1}{Z}\sum_{i=\pm1} \sigma^i_{A/B} e^{\beta\overline{h}\sigma^i_{A/B}} = \frac{2}{Z} sinh(\beta\overline{h}) = tanh(\beta\overline{h}) \tag{6}$$

using $c = 2^d = 4$as the number of close neighbors depending of the dimension and we define $T_c = \epsilon c$ and get (in next question (3) we prove this definition):

$$\begin{cases} M_A = tanh(\beta h - \beta\epsilon c M_B) \\ M_B = tanh(\beta h - \beta\epsilon c M_A) \end{cases} \tag{7}$$

For $T \simeq T_c$, we assume $M_A \simeq M_B \ll 1$ and therefore can expand $arctgh(x) \simeq x + \frac{1}{3}x^3$and with a little algebra we get:

$$Tarctgh(M_A) \simeq T(M_A + \frac{1}{3}M_A^3) \simeq TM_A + \frac{1}{3}T_c M_A^3 = h - T_c M_B \tag{8}$$

$$Tarctgh(M_B) \simeq T(M_B + \frac{1}{3}M_B^3) \simeq TM_B + \frac{1}{3}T_c M_B^3 = h - T_c M_A \tag{9}$$

By adding and subtracting the equations we get:

$$2h - T_c(M_A + M_B) = T(M_A + M_B) + \frac{T_c}{3}(M_A^3 + M_B^3) \tag{10}$$

$$T_c(M_A + M_B) = T(M_A - M_B) + \frac{T_c}{3}(M_A^3 - M_B^3) \tag{11}$$

From $M_s = \frac{1}{2}(M_A - M_B)$ and $M = \frac{1}{2}(M_A + M_B)$ we get $M_A = M_s + M$ and $M_B = M - M_s$, which gives:

$$M_A^3 = (M_s + M)^3 = M_s^3 + 3M_s^2 M + 3M_s M^2 + M^3 \tag{12}$$

$$M_B^3 = (M + M_s)^3 = M^3 - 3M^2 M_s + 3MM_s^2 - M_s^3 \tag{13}$$

adding and subtracting the equations gives:

$$M_A^3 + M_B^3 = 2M^3 + 6M_s^2 M \tag{14}$$

$$M_A^3 - M_B^3 = 2M_s^3 + 6M^2 M_s \tag{15}$$

by substitution of Eq. 14, 15 and M, M_s to Eq. 10, 11 we finally get

$$\begin{cases} a).\, h = M(T + T_c) + \frac{T_c}{3}(M^3 + 3M_s^2 M) \\ b).\, 0 = M_s(T - T_c) + \frac{T_c}{3}(M_s^3 + 3M^2 M_s) \end{cases} \tag{16}$$

By using this equations we can find M_s for $h = 0$ and $T \simeq T_c$:
for $T \simeq T_c$and $h = 0 \to M \ll 1$and therefore we can assume that

$M^2 \simeq 0$, by substracting this terms to Equ. b. we get

$$M_s = \sqrt{3\frac{T_c - T}{T}} \tag{17}$$

(3)
when $T = T_c$ the staggered magnitization equal to zero, $M_s = 0$, as can be seen

from the last result (Eq. 17) this result confirms our definition $T_c = \epsilon c$.

As for h$\neq 0$:

from Eq. 16.b we have:

$$M_s(\underbrace{T - T_c(1 - M^2)}_{\tilde{T_C}}) + \frac{1}{3}T_c M_s^3 = 0 \tag{18}$$

$$\tilde{T_c} = T_c(1 - M^2) = c\epsilon(1 - M^2) \tag{19}$$

for $T \simeq T_c$from Eq. 16.a we get:

$$h = 2T_c M + \frac{1}{3}T_c M^3 + M_s^2 T_c M \tag{20}$$

for $T \simeq T_c$ we can assume that M, $M_s \ll 1$ therefore $M_s^2 \to 0$ and $M^3 \to 0$

With these assumptions from Eq. 20 we get

$$M = \frac{h}{2T_c} \Rightarrow \tilde{T_c} = T_c(1 - \frac{h^2}{4T_c^2}) \tag{21}$$

(4) In antiferromagnet $M_s = 0$, $(T = 0)$, but if the magnetic field $h \neq 0$ at $T = 0$ we can steel use previous algebraic definitions like in ferromagnetic case:

$$\begin{cases} (T - T_c)M_s + \frac{1}{3}T_c(3M^2M_s + M^3) = 0 \\ (T + T_c)M + \frac{1}{3}T_c(3M_s^2M + M^3) = h \end{cases} \tag{22}$$

In antiferromagnetic case we have: $M_s = 0$, so $h = MT_c$, $T_c = \epsilon_c$

At $T = 0$ there is a definition of magnetization $M = mN$, in our case we have two sabblatices so in order do not count nearest neighbors twice:

$N \longrightarrow \frac{N}{2}$

so $h = T_c m \frac{N}{2}$, but in case of antiferromagnet that have $\uparrow\downarrow$ constraction

$m = -1$, so $h = -\frac{N}{2}T_c = -\frac{N}{2}\epsilon_c = h_c$ and this is an identification of ground state:

$E(\uparrow\downarrow\uparrow\downarrow) = -\frac{N}{2}\epsilon_c$, so for ddestroying the antiferromagnetic order needed $h < h_c$.

(5) we can define $M_A = M_{A0} + \delta M_A$ and $M_B = M_{B0} + \delta M_B$

$$M = M_A + M_B = M_{A0} + \delta M_A + M_{B0} + \delta M_B \tag{23}$$

by using the approximation:

$$tanh(x) = tanh(x_0 + \delta x) \simeq tanh(x_0) + \frac{\delta x}{cosh^2(x_0)} \tag{24}$$

and for small magnetic field we can define $\delta x = h\beta - \epsilon c\beta\delta M_B$ we get:

$$M_A = tanh(h\beta - \epsilon c\beta(M_{B0} + \delta M_B)) \simeq tanh(-\epsilon c\beta M_{B0}) + \frac{h\beta - \epsilon c\beta\delta M_B}{cosh^2(-\epsilon c\beta M_{B0})} \tag{25}$$

and for $\delta x = h\beta - \epsilon c\beta\delta M_A$ we get:

$$M_B = tanh(h\beta - \epsilon c\beta(M_{A0} + \delta M_A)) \simeq tanh(-\epsilon c\beta M_{A0}) + \frac{h\beta - \epsilon c\beta\delta M_A}{cosh^2(-\epsilon c\beta M_{A0})} \tag{26}$$

At temperature that much lower from the critical temperature we can assume that $M_{A0} = -M_{B0}$, and by attaching $M_A$and $M_B$we get an equation for the magnetism M:

$$M = M_A + M_B = \frac{2h\beta - \epsilon c\beta(\delta M_A + \delta M_B)}{cosh^2(\epsilon c\beta M_{A0})} \tag{27}$$

$2M_s = M_A - M_B = M_{A0} + \delta M_A - M_{B0} - \delta M_B = 2M_{A0} + (\delta M_A - \delta M_B) \simeq 2M_{A0}$

for $\delta M_A - \delta M_B \to 0$, and we get:

$$M \simeq \frac{2h\beta - \epsilon c\beta(\delta M_A + \delta M_B)}{cosh^2(\epsilon c\beta M_s)} \tag{28}$$

at low T we have very small M. therefore we can assume that $\delta M_A + \delta M_B = \delta M \simeq M$

by substitution and isolating M we have:

$M = \frac{h}{\frac{1}{2}T_c + T\cosh^2\left(\frac{TM_s}{T_c}\right)}$, where $T_c = \epsilon c, \quad T = \frac{1}{\beta}$

$\chi = \left(\frac{\partial M}{\partial h}\right)_{h\to 0} = \frac{1}{T\cosh^2\left(\frac{TM_s}{T_c}\right) + \frac{1}{2}T_c}$

(6)
let define linear approximation of suseptability in case of $T \approx T_c$ at two regions $T \geq T_c$ and $T \leq T_c$:

for $h = 0$: $\chi = \frac{1}{T_c + T\cosh^2\left(\frac{T_c M_s}{T}\right)}$

a) for $T \geq T_c$

Here $M_s = 0$ because the order parameter of antiferromagnet defined by a magnetization of lattices A and B, that in region of T>T$_c$ have zero values, so

$M_s = \frac{1}{2}(M_A - M_B) = 0$,where $M_A = M_B = 0$, than

$\cosh^2\left(\frac{T_c M_s}{T}\right) = 1$

$\chi = \frac{1}{T_c + T}$

$\frac{1}{\chi} = T + T_c$

b) for $T \leq T_c$,

In this region order parameter of antiferromagnet defined by $M_s = \sqrt{3\frac{T_c - T}{T_c}}$

we can use the approximation $\cosh^2 x \simeq \left(1 + \frac{1}{2}x^2\right)^2 \simeq 1 + x^2$

$\cosh^2\left(\frac{T_c M_s}{T}\right) \simeq 1 + \left(\frac{T_c}{T}\right)^2\left(3\frac{T_c - T}{T_c}\right) \simeq 1 + \left(3\frac{T_c - T}{T_c}\right)$

the last approximation we can do because $T_c \sim T$ and therefore$\left(\frac{T_c}{T}\right)^2 \to 1$

$\chi = \frac{1}{T_c + T\left(1 + \frac{T_c^2 M_s^2}{T^2}\right)} \sim \frac{1}{T_c + T\left(1 + \left(3\frac{T_c - T}{T_c}\right)\right)} = \frac{1}{4T_c - 2T}$

$\frac{1}{\chi} = 4T_c - T$