E5721: Antiferromagnet

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The problem: Antiferromagnet

Consider Ising model on a 2D lattice with antiferromagnetic interaction ($\epsilon = -\epsilon_0$). You can regard the lattice as composed of two sublattices A and B, such that $M = \frac{1}{2}(M_A + M_B)$ is the averaged magnetization per spin and $M_s = \frac{1}{2}(M_A - M_B)$ is the staggered magnetization

(a) Explain the claim: for zero field ($h = 0$), Ising antiferromagnet is the **same** as Ising ferromagnet, where M_s is the order parameter.
(write the expression for $M_s(T)$ at $T \approx T_c$, based on the familiar solution of the ferromagnet case)

(b) Given h and ϵ_0, find the coupled mean field equations for M_A and M_B.

(c) Find the critical temperature T_c for $h = 0$ and also for small h.
(Hint for $h = 0$: use the same procedure of expanding $arctanh(x)$ as in the ferromagnet case)
(Hint for small h: you may use the most extreme simplification that doesn't give the trivial solution)

(d) Find the critical magnetic field h_c above which the system no longer acts as an antiferromagnet at zero temperature (T=0).

(e) Find an expression for the susceptibility as a function of the staggered magnetization $\chi(M_s)$.

(f) In the region of $T \approx T_c$ give a linear approximation of $\frac{1}{\chi}$ as a function of the temperature T

The solution:

(a) The Hamiltonian as h=0:

$$H_{AF} = \epsilon_0 \cdot \sum_{<i,j>} \sigma_i \sigma_j$$

$$= -\epsilon \cdot \sum_{<i,j>} \sigma_i \sigma_j = H_F(h = 0)$$

$$M_s(T) = \left(3\frac{T_c - T}{T}\right)^{\frac{1}{2}}$$

(b) The Hamiltonian for lattice A is of the form:

$$H_A = \epsilon_0 \cdot \sum_{<i,j>} \sigma_i \sigma_j - h \sum_i \sigma_i$$

Where $< i, j >$ symbols close neighbors between lattice A to B and $h = \mu_B \cdot B$
Using $\gamma = 2^d = 4$ as the number of close neighbors depending of the dimension and

we define magnetism per spin (of lattice A/B):

$$< \sigma^{A/B} >= \frac{\tilde{M}_{A/B}}{\mu_B N_{A/B}} = M_{A/B} = \frac{1}{N_{A/B}} \cdot \sum_i \sigma_i^{A/B}$$

Let us develop H using the mean field theory:

$$H_A = \epsilon_0 \gamma \sum_i \sigma_i^A \cdot < \sigma^B > -h \sum_i \sigma_i$$
$$= \epsilon_0 \gamma \cdot N_A \cdot M_A \cdot M_B - h \cdot N_A \cdot M_A$$
$$= (\epsilon_0 \gamma \cdot M_B - h) \cdot N_A \cdot M_A$$

The Hamiltonian of one spin from sub lattice A:

$$H_i^A = (\epsilon_0 \gamma \cdot M_B - h) \cdot \sigma_i$$

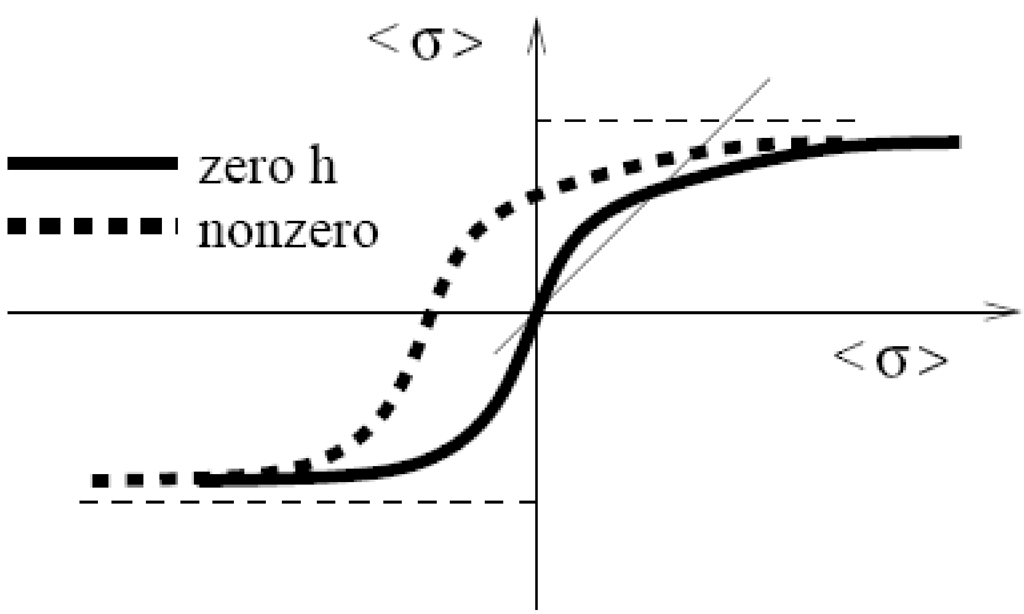
$$Z_1^A = \sum_{\sigma_i^A} e^{-\beta \cdot H_i^A} = 2cosh(\frac{1}{T} \cdot (\epsilon_0 \gamma \cdot M_B - h))$$

$$(I) M_A = -T \cdot \frac{\partial ln(Z_i^A)}{\partial h} = tanh(\frac{1}{T}(h - \epsilon_0 \gamma \cdot M_B))$$
$$(II) M_B = tanh(\frac{1}{T}(h - \epsilon_0 \gamma \cdot M_A))$$

(c) At zero magnetic field $(h = 0)$:

$$M_{A_0} = -M_{B_0} \quad \rightarrow \quad M_{A_0} = tanh\left(\frac{\epsilon_0 \gamma M_{A_0}}{T}\right)$$

This equation should be solved for M_{A_0}.



By inspection of the plot we observe that for h = 0 the condition for getting a non trivial solution is $\gamma \cdot \epsilon_0 / T > 1$. Therefore

$$(c_1)\ \mathbf{T_c} = \gamma \cdot \epsilon_\mathbf{0}$$

Lets use the solution of (b) and search for an expression for M and M_s:

$$\begin{cases} (I)\ \frac{1}{T}(h - T_c \cdot M_B) = arctanh(M_A) \approx M_A + \frac{1}{3}M_A^3 \\ (II)\ \frac{1}{T}(h - T_c \cdot M_A) = arctanh(M_B) \approx M_B + \frac{1}{3}M_B^3 \end{cases}$$

$$\begin{cases} (I)\ h = T_c \cdot M_B + T \cdot M_A + \frac{1}{3}T \cdot M_A^3 \\ (II)\ h = T_c \cdot M_A + T \cdot M_B + \frac{1}{3}T \cdot M_B^3 \end{cases}$$

$$\begin{cases} (I) + (II):\ 2h = (T + T_c) \cdot (M_A + M_B) + \frac{1}{3}T(M_A^3 + M_B^3) \\ \qquad \approx (T + T_c) \cdot (M_A + M_B) + \frac{1}{3}T_c(M_A^3 + M_B^3) \\ (I) - (II):\ 0 = T_c \cdot (M_B - M_A) + T \cdot (M_A - M_B) + \frac{T}{3}(M_A^3 - M_B^3) \end{cases}$$

After some calculations [and using $M_A \cdot M_B = m^2 - M_s^2$] we get:

$$\begin{cases} (I) + (II):\ h = (T + T_c + T_c \cdot M_s^2) \cdot M + \frac{1}{3}T_c \cdot M^3 \\ (I) - (II):\ 0 = (T - T_c + T \cdot M^3) \cdot M_s + \frac{T}{3}M_s^3 \end{cases}$$

Which gives the following solutions:

$$(1)\ M_s = 0 \quad or \quad 3 \cdot \left(\frac{T_c - T}{T}\right) = 3M^2 + M_s^2$$

$$(2)\ T \approx T_c\ \mapsto\ \frac{h}{T_c} = (2 + M_s^2) \cdot M + \frac{1}{3}M^3$$

As expected from the second equation we get M = 0 in the absence of an external field, and from the first equation we get the order parameter $M_s(T) = \left(3 \cdot \frac{T_c - T}{T}\right)^{\frac{1}{2}}$, which satisfies the same equation as in the ferromagnetic problem.

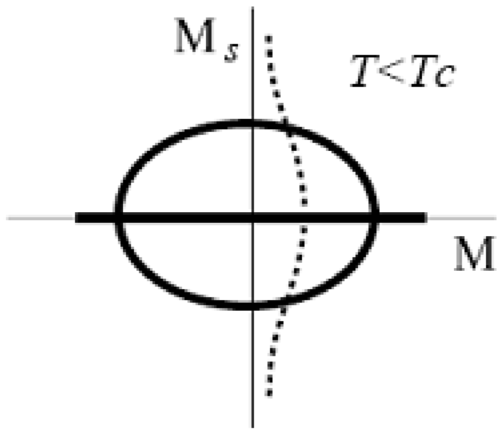
We will show that if we switch on the magnetic field, T_c is shifted to a lower temperature.
Assuming small magnetic field we can neglect M^3 and M_s at equation (2) and get:

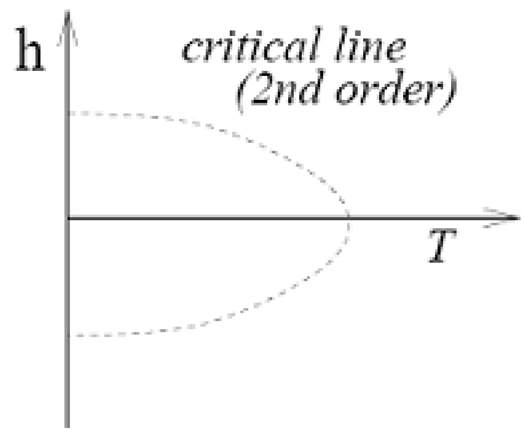
$$M = \frac{h}{2T_c^{(0)}}$$

Putting this result at equation (1) we get:

$$M_s^2 = 3\left(\frac{T_c^{(0)} - \frac{h^2}{4T_c^{(0)}} - T}{T_c^{(0)}}\right) \equiv 3\left(\frac{T_c - T}{T}\right)$$

Where we got $T_c = T_c^{(0)} - \frac{h^2}{4T_c^{(0)}}$





(d) We can now express the critical magnetic field from the critical temperature:

$$h_c^2 = 4\epsilon^2\gamma^2 - 4\epsilon\gamma T_c(h)$$

$$\Rightarrow \quad h_c = 2\sqrt{\epsilon^2\gamma^2 - \epsilon\gamma T_c(h)}$$

(e) Let us define:

$$M_A = M_{A_0} + \delta M_A \ ; \ M_B = M_{B_0} + \delta M_B = -M_{A_0} + \delta M_B$$

$$M = M_A + M_B = \delta M_A + \delta M_B$$

By using the approximation:

$$tanh(x) = tanh(x_0 + \delta x) \approx tanh(x_0) + \frac{\delta x}{cosh^2(x_0)}$$

We get:

$$M_A = tanh\left(-\frac{T_c}{T}M_B + \frac{1}{T}\cdot h\right) = \left(-\frac{T_c}{T}(M_{B_0} + \delta M_B) + \frac{1}{T}\cdot h\right)$$

$$\approx \underbrace{tanh\left(\frac{T_c}{T}M_{B_0}\right)}_{M_{A_0}} + \underbrace{\frac{-T_c\cdot \delta M_B + h}{T\cdot cosh^2(-\frac{T_c}{T}M_{B_0})}}_{\delta M_A}$$

And in the same way we can get:

$$\delta M_B = \frac{-T_c\cdot \delta M_A + h}{T\cdot cosh^2(-\frac{T_c}{T}M_{A_0})} = \frac{-T_c\cdot \delta M_A + h}{T\cdot cosh^2(\frac{T_c}{T}M_{B_0})}$$

And by attaching the two equations we get an equation for the magnetism M:

$$M = \frac{1}{2}(\delta M_A + \delta M_B) = \frac{-\frac{T_c}{2}\cdot \delta M + h}{T\cdot cosh^2(\frac{T_c}{T}M_{B_0})}$$

After isolating M we have:

$$M = \frac{h}{T_c + T \cdot cosh^2(\frac{T_c}{T} M_{B_0})}$$

$$M_s(h=0) = M_s(T) = \frac{1}{2}(M_{A_0} - M_{B_0}) = -M_{B_0}$$

$$\rightarrow \quad M = \frac{h}{T_c + T \cdot cosh^2(\frac{T_c}{T} M_s(T))}$$

Let us find the susceptibility:

$$X = \left(\frac{\partial M}{\partial h}\right)_{h \to 0}$$

$$\Rightarrow \quad X = \frac{1}{T_c + T \cdot cosh^2(\frac{T_c}{T} M_s(T))}$$

(f) Let us examine the susceptibility near T_c at two cases:

$\underline{T \ > \ T_c \ :}$

at that case the system is not an antiferromagnet yet so $M_s(T) = 0 \ \rightarrow \ cosh(\frac{T_c}{T} M_s(T)) = 1$ which gives us:

$$X_{T>T_c} = \frac{1}{T + T_c}$$

$$\Rightarrow \frac{1}{X_{T>T_c}} = T + T_c$$

$\underline{T \ < \ T_c \ :}$

By using the approximation:

$$cosh^2(y) \approx (1 + \frac{1}{2}y^2)^2 \approx 1 + y^2$$

$$X_{T<T_c} = \frac{1}{T + T \cdot \left(\frac{T_c}{T} M_s(T)\right)^2 + T_c}$$

A reminder:

$$M_s^2(T) = \left(3\frac{T_c - T}{T}\right)$$

And after substitution of $M_s(T)$ and taking into account that where $T \approx T_c \ \rightarrow \ \frac{T_c}{T} \approx 1$ we get:

$$X_{T<T_c} = \frac{1}{T + 3 \cdot T\left(\frac{T_c - T}{T}\right) + T_c}$$

$$X_{T<T_c} = \frac{1}{4T_c - 2T}$$

$$\frac{1}{X_{T<T_c}} = 4T_c - 2T$$

for conclusion we got:

$$\frac{1}{X} = \begin{cases} T + T_c \ ; \ T > T_c \\ 4T_c - 2T \ ; \ T < T_c \end{cases}$$