Exercises in Statistical Mechanics

This exercises pool is intended for a graduate course in "statistical mechanics". Some of the problems are original, while other were assembled from various undocumented sources. In particular some problems originate from exams that were written by B. Horovitz (BGU), S. Fishman (Technion), and D. Cohen (BGU).

===== [Exercise 5716]

Ferromagnetism for cubic crystal

A cubic crystal which exhibits ferromagnetism at low temperature, can be described near the critical temperature T_c by an expansion of a Gibbs free energy

$$G(\mathbf{H}, T) = G_0 + \tfrac{1}{2} r \mathbf{M}^2 + u \mathbf{M}^4 + v \sum_{i=1}^{3} M_i^4 - \mathbf{H} \cdot \mathbf{M}$$

where $\mathbf{H} = (H_1, H_2, H_3)$ is the external field and $\mathbf{M} = (M_1, M_2, M_3)$ is the total magnetization; $r = a(T - Tc)$ and G_0, a, u and v are independent of $\mathbf{H}$ and $T, a > 0, u > 0$. The constant v is called the cubic anisotropy and can be either positive or negative.

(a) At $\mathbf{H} = 0$, find the possible solutions of $\mathbf{M}$ which minimize G and the corresponding values of $G(0, T)$ (these solutions are characterized by the magnitude and direction of $\mathbf{M}$. Show that the region of stability of G is $u + v > 0$ and determine the stable equilibrium phases when $T < T_c$ for the cases (i) $v > 0$, (ii) $-u < v < 0$.

(b) Show that there is a second order phase transition at $T = T_c$, and determine the critical indices α, β and γ for this transition, i.e. $C_{V,H=0} \sim |T - T_c|^{-\alpha}$ for both $T > T_c$ and $T < T_c$, $|\mathbf{M}|_{H=0} \sim (Tc - T)^{\beta}$ for $T < T_c$ and $\chi_{ij} = \partial M_i / \partial H_j \sim \delta_{ij} |T - T_c|^{-\gamma}$ for $T > T_c$.