E5660: Ising model of absorption sites

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Question
M absorption sites are arranged along a ring.
The sites are fermionic, i.e number of particles absorbed at site i is $n_i = 0, 1$.
Between every two absorption sites a spin $\sigma_i = \pm 1$ is located. The ring is surrounded by gas in temperature T and chemical potential μ.
The absorption energy is $\epsilon > 0$ if the two adjacent spins are in the same direction, and $-\epsilon < 0$ otherwise.

1. Write an expression for the energy $E[\sigma_i, n_i]$ of a given conguration.
2. Calculate the partition function $\mathbb{Z}(\beta, \mu)$ using the transfer matrix method, write what is $T_{\sigma_i,\sigma_{i+1}}$in this problem.
3. Find Helmholtz function $F(T, \mu)$ assuming $M \gg 1$.
4. Write an expression for the number of adsorbed particles $N = \sum_i < n_i >$ as a function of β, μ.
5. Write an expression for the correlation length ξ that characterizes the spins in the system.

Solution
1. The energy

$$E[\sigma_i, n_i] = -\epsilon n_i \sigma_i \sigma_{i+1} - \mu n_i \tag{1}$$

2. The partition function

$$\mathbb{Z} = \sum_{\sigma_i, n_i} e^{-\beta E[\sigma_i, n_i]} = \sum_{\sigma_i = \pm 1} 1 + e^{-\beta(-\epsilon \sigma_i \sigma_{i+1} - \mu)} \tag{2}$$

defining the transfer matrix

$$T_{\sigma_i,\sigma_{i+1}} = 1 + e^{\beta \epsilon \sigma_i \sigma_{i+1} + \beta \mu} \tag{3}$$

$$T_{\sigma_i,\sigma_{i+1}} = \begin{bmatrix} 1 + e^{\beta\epsilon + \beta\mu} & 1 + e^{-\beta\epsilon + \beta\mu} \\ 1 + e^{-\beta\epsilon + \beta\mu} & 1 + e^{\beta\epsilon + \beta\mu} \end{bmatrix} \tag{4}$$

Notice that $T_{\sigma_i,\sigma_{i+1}}$is of the form of $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$, it is straightforward to find its eigenvalues

$$\lambda_\pm = (1 + e^{\beta\epsilon + \beta\mu}) \pm (1 + e^{-\beta\epsilon + \beta\mu}) \tag{5}$$

$$\lambda_+ = 2 + 2e^{\beta\mu} \cosh(\beta\epsilon) \quad \lambda_- = 2e^{\beta\mu} \sinh(\beta\epsilon) \tag{6}$$

one can write the partition function using the transfer matrix

$$\mathbb{Z} = \sum_{\sigma_1} \cdots \sum_{\sigma_M} < \sigma_1 | T | \sigma_2 > \cdots < \sigma_M | T | \sigma_1 > =$$

$$< \sigma_1 | T^M | \sigma_1 > = \lambda_+^M + \lambda_-^M \sim \lambda_+^M \tag{7}$$

3. The Helmholtz function

$$F = -\frac{1}{\beta} \ln(\mathbb{Z}) = -\frac{1}{\beta} [M \ln(2 + 2e^{\beta\mu} \cosh(\beta\epsilon))] \tag{8}$$

4. The number of absorbed particles

$$N = -\frac{\partial F}{\partial \mu} = M \frac{\cosh(\beta\epsilon)}{e^{-\beta\mu} + \cosh(\beta\epsilon)} \tag{9}$$

5. The correlation length

$$e^{-\frac{1}{\xi}} = \frac{\lambda_-}{\lambda_+} = \frac{\sinh(\beta\epsilon)}{e^{-\beta\mu} + \cosh(\beta\epsilon)} \tag{10}$$