$\boxed{C24}$ (HW 2009 9.3) <u>1d Ising + impurities</u>

$$\mathcal{H} = -J\sum_{i=1}^{N} \sigma_i \sigma_{i+1}(1-\lambda n_i) \qquad (n_i = 0,1)$$

$$N' = \sum_{i=1}^{N} n_i < N$$

$$\langle n_i \rangle = \frac{N'}{N}$$

a) $$Z\{n_i\} = \sum_{\{\sigma_i\}} e^{-\beta\hat{\mathcal{H}}} = \mathrm{Tr}(P\cdot P\cdot P \cdots P'\cdot P'\cdot P' \cdots P)$$

$$P = \begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J}\end{pmatrix} \qquad P' = \begin{pmatrix} e^{\beta J(1-\lambda)} & e^{-\beta J(1-\lambda)} \\ e^{-\beta J(1-\lambda)} & e^{\beta J(1-\lambda)}\end{pmatrix}$$

but $[P,P']=0$ they commute ie diagonalized by the same transform $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

thus we commute them to get

$$Z\{n_i\} = \mathrm{Tr}\left[P^{N-N'}\cdot(P')^{N'}\right] = \mathrm{Tr}\left[\begin{pmatrix}\lambda_1^{N-N'} & 0 \\ 0 & \lambda_2^{N-N'}\end{pmatrix}\begin{pmatrix}\lambda_1'^{N-N'} & 0 \\ 0 & \lambda_2'^{N-N'}\end{pmatrix}\right]$$

$$\lambda_1 = 2\cosh(\beta J) \qquad \lambda_1' = 2\cosh(\beta J(1-\lambda))$$

$$\lambda_2 = 2\sinh(\beta J) \qquad \lambda_2' = 2\sinh(\beta J(1-\lambda))$$

for $\lambda_1 > \lambda_2$, $\lambda_1' > \lambda_2'$

$$Z\{n_i\}\Big|_{N,N'\to\infty} = (2\cosh(\beta J))^{N-N'}\cdot(2\cosh(\beta J(1-\lambda)))^{N'}$$

$$Z = \binom{N}{N'} Z\{n_i\} \to \boxed{F = -kT\ln\binom{N}{N'} - kT(N-N')\ln(2\cosh(\beta J)) - kTN'\ln(2\cosh(\beta J(1-\lambda)))}$$

b)

The disorder-averaged F cannot depend on $\{n_i\}$ (the configuration is static & doesn't contribute to the entropy)

$$\langle F\rangle = -kT(N-N')\ln(2\cosh\beta J) - kTN'\ln(2\cosh(\beta J(1-\lambda)))$$

$$\Delta S = \frac{-\partial F}{\partial T} + \frac{\partial\langle F\rangle}{\partial T} = k_B\ln\binom{N}{N'} = k_B N\ln N - k_B(N-N')\ln(N-N') - k_B N'\ln N'$$

$$= k_B N\ln\frac{N}{N-N'} - k_B N'\ln\frac{N'}{N-N'} > 0$$