C28

1D Potts model

N atoms, each with p states, 1d chain

$$\mathcal{H} = -J \sum_{i=1}^{n} \sum_{i_n=1}^{p} \delta_{i_n, i_{n+1}}$$

$$i_{N+1} = i_1$$

a) Free energy for $p=2$ $(N \to \infty)$

b) Free energy for general p.

c) Internal energy ... at low and high T limits.

$$T_{ij} = \begin{pmatrix} e^{\beta J} & 1 & & 1 \\ 1 & e^{\beta J} & & \\ & & \ddots & \\ 1 & & & \end{pmatrix}$$

$$Z = \mathrm{tr}\,(e^{-\beta \mathcal{H}}) = \sum_{\sigma_1=1}^{p} \langle \sigma_1 | e^{-\beta \mathcal{H}} | \sigma_1 \rangle$$

$$= \sum_{\sigma_1} \sum_{\sigma_2} \sum_{\sigma_3} \langle \sigma_1 | P_{12} | \sigma_2 \rangle \langle \sigma_2 | P_{23} | \sigma_3 \rangle$$

$$= \mathrm{tr}\left((P_{ij})^N\right) \approx \mathrm{tr}\ \lambda_{max}^N \approx \lambda_{max}^N$$

$$p = 2 \rightarrow \begin{vmatrix} e^{\beta J} - \lambda & 1 \\ 1 & e^{\beta J} - \lambda \end{vmatrix} = 0 \rightarrow (e^{\beta J} - \lambda)^2 - 1 = 0$$

$$\lambda = e^{\beta J} \pm 1$$

choose largest eigenvector

$$F = -kTN \ln \lambda_{max} = -kTN \ln (e^{\beta J} + 1) = -kTN \ln \left[(e^{\frac{\beta J}{2}} + e^{-\frac{\beta J}{2}}) e^{\frac{\beta J}{2}}\right]$$

$$= -kTN \ln \left(2 \cosh \frac{\beta J}{2}\right) - \frac{N}{2} J$$

similar to Ising model

$$\begin{matrix} \text{— } J \\ \text{— } 0 \end{matrix} \rightarrow \begin{matrix} \text{— } +\frac{J}{2} \\ \text{— } -\frac{J}{2} \end{matrix} + \frac{J}{2}$$

b) for general vector

$$\hat{T}_{ij} \begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix} = \lambda \begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix}$$

$$\rightarrow \forall a_i: \left(e^{\beta J} a_i + 1 \cdot \sum_{j \neq i} a_j = \lambda a_i \right)$$

$$e^{\beta J} a_i + \sum_{\forall j} a_j - a_i = \lambda a_i$$

$$a_i \left(\lambda + 1 - e^{\beta J} \right) = \sum_{\forall j} a_j = \text{Const} \quad (\text{doesn't depend on } i)$$

$\sum_{\forall i}$ ניתן לראות שכל אגף תלוי ב-i לכן גם אפשר לסכום

($\sum_{\forall i} 1 = p$ [illegible])

$$\left(\sum_{\forall i} a_i \right) \left(\lambda + 1 - e^{\beta J} \right) = p \left(\sum_{\forall j} a_j \right)$$

$\sum a_i = 0$

$\forall a_i = 0$ or $\left(\lambda + 1 - e^{+\beta J} \right) = p \neq 0$ or $\lambda + 1 - e^{+\beta J} = 0$

$$\boxed{\lambda_{max} = e^{\beta J} + p - 1}$$

$\lambda_{other} = e^{+\beta J} - 1 < \lambda_{max}$

degenerate eigvals

$$F = -kTN \ln \left(e^{\beta J} + p - 1 \right)$$

E) $E = -\frac{\partial \ln Z}{\partial \beta} = \frac{\partial}{\partial \beta} (\beta F) =$

$$= -N \frac{e^{\beta J} J}{e^{\beta J} + p - 1} = \begin{cases} -\frac{J}{p} N & T \to \infty, \beta \to 0 \\ -JN & T \to 0, \beta \to \infty \end{cases}$$

At $T=0$ all atoms in the same state as there is no entropy gain.

At $T \to \infty$ entropy wins & we have atoms equally distributed so that Prob. that n.n. is of same spin is $\frac{1}{p}$