Exercises in Statistical Mechanics

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This exercises pool is intended for a graduate course in "statistical mechanics". Some of the problems are original, while other were assembled from various undocumented sources. In particular some problems originate from exams that were written by B. Horovitz (BGU), S. Fishman (Technion), and D. Cohen (BGU).

═══ [Exercise 5641]

Ising with long range interaction

Consider a cluster of N spins $s_i = \pm 1$. The interaction between *any* two spins is $-\epsilon s_i s_j$, with $\epsilon > 0$. The interaction of each spin with the external magnetic field is $-\mathsf{H} s_i$. The total magnetization is defined as $\mathsf{m} = \sum s_i$. The inverse temperature is β.

(a) Show that the partition function can be written as $Z(\beta, \mathsf{H}) = \sum_{\mathsf{m}} g(\mathsf{m}) \exp\left(\frac{1}{2}\mathsf{B}m^2 + h\mathsf{m}\right)$. Express $g(\mathsf{m})$ and B and h using $(N, \epsilon, \beta, \mathsf{H})$.

(b) Assume that $\mathsf{B} = b/N$, and define the magnetization as $M = \mathsf{m}/N$. Write the partition function as $Z(b, h) = \sum_M \exp(-N\mathcal{A}(M))$. Write the expressions for $\mathcal{A}(M)$ and for its derivatives $\mathcal{A}'(M)$ and $\mathcal{A}''(M)$.

(c) Determine the critical temperature T_c, and write an equation for the mean field value of M. Make a qualitative plot of $\mathcal{A}(M)$ below and above the critical temperature.

(d) Write an approximation for $\mathcal{A}(M)$ up to order M^4. On the basis of this expression determine the temperature range where mean filed theory cannot be trusted. Hint: you have to estimate the variance $\langle M^2 \rangle$ in the Gaussian approximation. What happens with this condition in the thermodynamic limit $(N \to \infty)$?

(e) Find an expression for the heat capacity in the mean field and in the Gaussian approximations.